Cleaning Up Your Table

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1 Introduction

Environment influences productivity: better environment implies higher productivity, worse environment implies lower productivity. It is profitable then to improve the environment. But at the same time improving the environment takes the resources away from the production. So, it can not be optimal to focus solely on the environment. Neither it is optimal to focus solely on the production, as production pollutes the environment, thus decreasing the productivity.

This problem applies to broad range of more specific economic situations. If to give an example, consider R&D activities of a firm. Investing into R&D increases the productivity and competitiveness but takes money and people away from the production. At the same time devoting the resources solely to the production will relatively decrease the productivity and competitiveness of the firm if the other competing firms invest in R&D at the time.

There are two popular solutions to this trade-off between environment and production. The first one is to switch between periods of environmental improvements and periods of production, where the switch occurs when the environment is sufficiently good or when it is intolerably bad. The second solution is to permanently keep the environment in a good state, what implies some even distribution of resources between environment and production.

Next I set up a simple dynamic optimisation problem to find what is optimal indeed.[1]

2 Setup

The notation is:

$x(t)$ - productive environment,

$y(t)$ - production,

$u(t)$ - investments into the environment,

$v(t)$ - inputs into the production.

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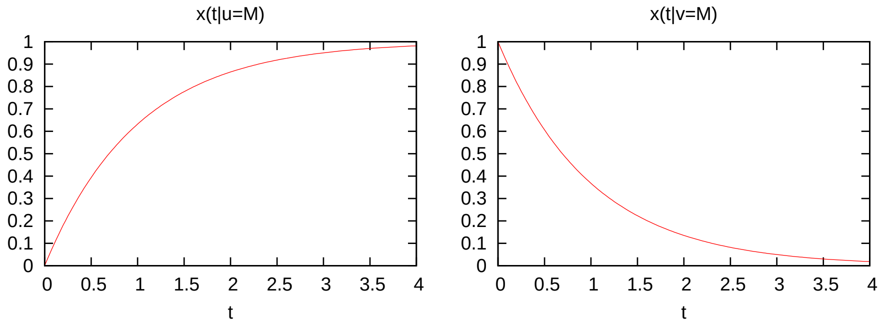
[1]Personally I tend to use the first solution.

Figure 1: Dynamics of the environment in the cases of improvements and pollution



There is a degree of freedom in defining the environment, so let me talk about *productive* environment x in a sense that

$$y = xv. \tag{1}$$

The total resources are limited, so

$$u(t) + v(t) \leq M \tag{2}$$

where M is constant over time. I also assume that disinvestments are not available:

$$u \geq 0, \quad v \geq 0. \tag{3}$$

Either if all the resources are invested into the environment ($u = M$) or if they all are used in the production ($v = M$), I would like the environment $x(t)$ to exhibit certain pattern over time. The required pattern is given in fig. 1. The rationale is as follows. If all the resources are invested into the environment and none are used in the polluting production, then the productive environment should grow. Moreover, the first steps of environmental improvements usually treat the most critical issues, so the productive environment should grow more sharply at the beginning, later steps just add some "polish" thus improving the productive environment only slightly. I also assume it is not possible to raise the productive environment to any extent you want, i.e. there is an upper limit.

If all the resources are used in the polluting production and nothing is invested into the environment, the productive environment should decline. Moreover, as production pollutes exactly those parts of the environment, where it operates, the productive environment should decline more sharply at first. Later on, if the productive environment is already bad, then extra pollution shouldn't matter that much. I also assume a lower limit here, i.e. however bad the environment is, it is still possible to produce at least something, so the productive environment is above zero.

A simple process for x satisfying these assumptions is

$$\dot{x} = (1 - x)u - kxv \tag{4}$$

where k stands for the strength of pollution. I assume $0 < k < 1$, that means I consider only cases of "moderately" polluting productions.

Finally, the objective is to maximise total production over time period $[0, T]$:

$$\int_0^T y(t)dt \to \max_{u(t),v(t)} \tag{5}$$

given an initial production environment x_0 and allowing for any production environment at the end ($x(T)$ - free).

Obviously, $u + v < M$ is not optimal, this gives $v = M - u$ and the problem can be summarised as follows

$$\int_0^T x(M-u)dt \to \max_{0 \le u \le M}, \tag{6}$$

$$\dot{x} = (1-x)u - kx(M-u), \tag{7}$$

$$x(0) = x_0, \quad x(T) \text{ - free.} \tag{8}$$

3 Solution

The problem (6)-(8) is an optimal control problem. I apply Pontryagin maximum principal to solve it. Given the Hamiltonian

$$H = x(M-u) + p((1-x)u - kx(M-u)) \tag{9}$$

the necessary conditions are

$$\max_{0 \le u \le M} H \iff \max_{0 \le u \le M} \left((-x + p(1 - x + kx))u + xM(1 - pk)\right), \tag{10}$$

$$H_p = \dot{x} \iff \dot{x} = (1-x)u - kx(M-u), \tag{11}$$

$$H_x = -\dot{p} \iff M - u + p(-u - k(M-u)) = -\dot{p}. \tag{12}$$

and the boundary and transversality conditions are

$$x(0) = x_0, \tag{13}$$

$$p(T) = 0. \tag{14}$$

Note that (11) together with $0 < k < 1$ and $0 \le u \le M$ imply that $0 < x < 1$.

Next, let

$$f = -x + p(1 - x + kx) \tag{15}$$

and consider the cases for $f > 0$, $f < 0$ and $f = 0$.

3.1 $f > 0$

If $f > 0$ then $u = M$, so

$$\dot{x} = (1-x)M, \tag{16}$$

$$\dot{p} = pM. \tag{17}$$

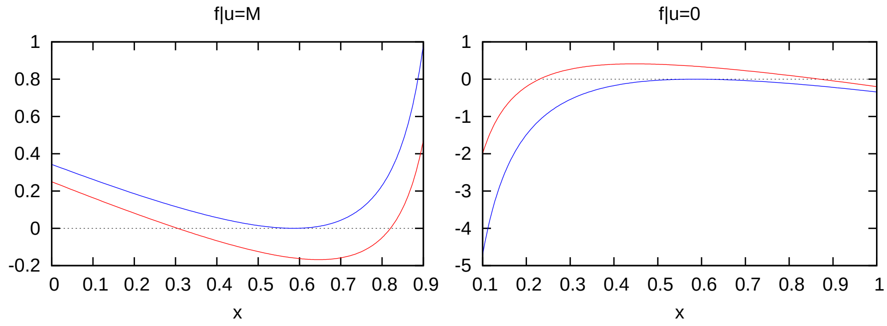
Taking into account that $0 < x < 1$ it gives

$$x = 1 - e^{-Mt + A_1}, \tag{18}$$

$$p = B_1 e^{Mt} = \frac{C_1}{1-x} \tag{19}$$

with $C_1 = e^{A_1} B_1$.

Figure 2: Functions f for $u = M$ and $u = 0$



3.2 $f < 0$

If $f < 0$ then $u = 0$, so

$$\dot{x} = -kxM, \tag{20}$$

$$\dot{p} = kMp - M. \tag{21}$$

Taking into account that $0 < x < 1$ it gives

$$x = e^{-kMt+A_2}, \tag{22}$$

$$p = B_2 e^{kMt} + \frac{1}{k} = \frac{C_2}{x} + \frac{1}{k} \tag{23}$$

with $C_2 = e^{A_2} B_2$.

3.3 $f = 0$

If $f = 0$, then any $u(t)$ such that $0 \leq u(t) \leq M$ is allowed. But from (15) it also follows that

$$p = \frac{x}{1 - x + kx}. \tag{24}$$

Substituting this expression into (12), using the expression for $\dot{x}$ and rearranging gives

$$(1-k)x^2 - 2x + 1 = 0. \tag{25}$$

Given that $0 < x < 1$ the solution is

$$x = \frac{1 - \sqrt{k}}{1 - k}. \tag{26}$$

This implies that $x(t)$ is constant, so $p(t)$ is constant, so $\dot{p} = 0$ and thus finally

$$u = \frac{kx}{1 - x + kx} M. \tag{27}$$

As required, $0 \leq u \leq M$.

Let us consider the cases $f > 0$ and $f < 0$ in more detail. If $f > 0$ at some time $t = \tau$, then from (15) and (19) it follows that

$$f = -x + C_1 k \frac{x}{1-x} + C_1. \tag{28}$$

As $0 < x < 1$ and $f(x(\tau)) > 0$ it is required that $C_1 > 0$. Plots of $f(x)$ in this case for different C_1 are given in fig. 2, left part. The minimum is approached at $x^* = 1 - \sqrt{C_1 k}$. If $f(x^*) = 0$, then

$$x^* = \frac{1-\sqrt{k}}{1-k}. \tag{29}$$

If $f < 0$ at some time $t = \tau$, then from (15) and (23) it follows that

$$f = \left(\frac{C_2}{x} + \frac{1}{k}\right)(1-x) + C_2 k. \tag{30}$$

As $0 < x < 1$ and $f(x(\tau)) < 0$ it is required that $C_2 < 0$. Plots of $f(x)$ in this case for different C_2 are given in fig. 2, right part. The maximum is approached at $x^* = \sqrt{-C_2 k}$. If $f(x^*) = 0$, then

$$x^* = \frac{1-\sqrt{k}}{1-k} \tag{31}$$

as well.

Now everything is prepared to discuss the solution. I start from what can not be.

Lemma 1. *In general it is not optimal to switch between periods of sole production and periods of sole environmental improvements.*

Proof. Assume $x_0 < x^* = \frac{1-\sqrt{k}}{1-k}$. If to start from $u = 0$, $x(t)$ decreases in time according to (22). Then for any $C_2 < 0$, $f(x)$ decreases – see the right plot in fig. 2. So, there will be no switching later on. But as $C_2 < 0$ so is B_2 (from (23)), this means p is also decreasing over time. If T is not small enough, i.e. in a general case, it will not be possible to achieve $p(T) = 0$, which is a necessary condition for the optimality.

If to start from $u = M$, $x(t)$ increases in time according to (18). Then $f(x)$ decreases at first – see the left plot in fig. 2. If $\min_{0<x<1} f(x) > 0$, then f will start increasing later on and there will be no switching. As $C_1 > 0$, so is B_1 and thus $p > 0$ always. This can not be optimal.

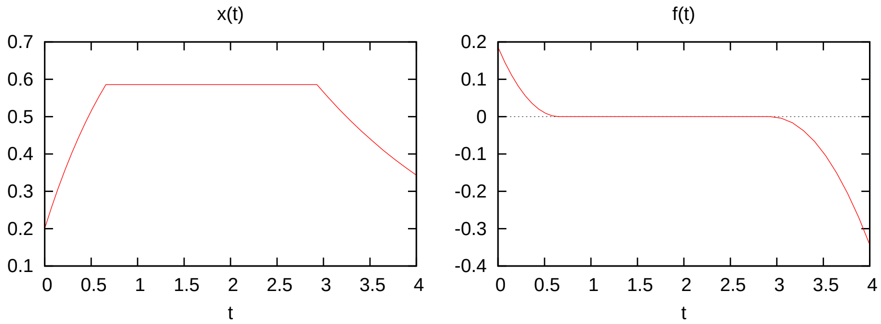
Let f cross zero at some x. If $f = 0$ exactly at $x = x^*$, then it is not possible to switch to $u = 0$ as $f(x+0) > 0$. Otherwise, if switching to $u = 0$ occurs, then it occurs at $x < x^*$. As there was switching to $u = 0$, x starts decreasing, and as the switching occurred at $x < x^*$, the new f start decreasing as well leading to no more switches later on. As before, this can lead to a solution only for T small enough.

Analogues arguments hold for $x_0 > x^*$ and $x_0 = x^*$. □

Lemma says that for T above a certain threshold switching between $u = M$ and $u = 0$ can not lead to a solution. It can also be reformulated in a way that in a solution switching between $u = M$ and $u = 0$ can not occur more than once.

Precisely the same way of reasoning gives the solution. So I state it bellow without a proof.

Figure 3: A typical solution for $x_0 < x^*$



Lemma 2. *In general it is optimal to achieve x^* at first, starting with $u = M$ if $x_0 < x^*$ and with $u = 0$ if $x_0 > x^*$. Then the environment should be kept at constant level x^* with the effort distributed between both production and environment according to* (27)*. Finally, in advance of T there should be a switch to $u = 0$ as to approach $p = 0$ at time T.*

An typical solution for $x_0 < x^*$ is given in fig. 3.[2]

4 Comments

I would like to give two comments. The first one is that the optimal level of environment x^*, as given by (26), negatively depends on k. So, the stronger is the pollution, the lower is the level of environment, which is optimal to sustain.

The second comment questions why switching between periods of production and periods of environmental improvements is observed. In the model presented, it may occur for small T, when it indeed becomes optimal to increase environment at first and then to concentrate on production. Then, for example, if an economic agent divides his future activities into projects and concentrate every time only on a current project without future environmental concerns, regular switches between good and bad environment will be observed.

[2] $x_0 = 0.2$, $k = 0.5$, $M = 1$, $T = 4$