Oligopolistic competition in price and quality

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Abstract

We consider an oligopolistic market where firms compete in price and quality and where consumers have heterogeneous information: some consumers know both the prices and quality of the products offered, some know only the prices and some know neither. We show that two types of signalling equilibria are possible. Both are characterized by dispersion and Pareto-inefficiency of the price/quality offers. Better price/quality combinations are signalled with lower prices in one type and with higher prices in the other type.

Key Words: oligopoly, competition, price, quality, imperfect information, signalling.

JEL Classification: D43, D83, L13, L15.

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1 Introduction

The very nature of competition implies that firms compete in as many ways as possible, and not just in price. Firms will choose that combination of strategic variables that serves their interests best. However, if consumers are homogeneous in their preferences among these variables and if they are fully informed about all relevant product characteristics, then this multi-dimensional form of competition can be expressed in terms of a one dimensional competition model, essentially identical in nature to that of price competition.

When consumer preferences differ, or when some consumers are better informed than others, the competitive process involving many dimensions does not have a single dimension analogue and should be analysed in its own right. In this paper we restrict our attention to markets where firms compete in two dimensions: price and quality. There are different approaches known in the literature dealing with endogenous price/quality competition. Chan and Leland (1982), Cooper and Ross (1984), and Schwartz and Wilde (1985) emphasize the heterogeneity of information among consumers. Wolinsky (1983), Rogerson (1988), and Besancenot and Vranceanu (2004) additionally emphasize the heterogeneity of preferences. Another dimension along which these models differ is the type of market interaction considered. The above models either consider perfect competition or monopolistic competition.

Contrary to the aforementioned literature, we address the issue of price/quality competition in a *strategic* oligopoly model where price and quality are endogenously chosen and concentrate on the role of consumers having heterogeneous information (and therefore take consumer preferences to be identical). In a recent paper Armstrong and Chen (2009) consider a similar framework where they focus on boundedly rational consumers that observe the prices but do not infer the corresponding quality, even if such inference is possible. We, on the other hand, focus on rational consumers, and the signalling role of prices for quality inference plays a central role in our study.

We ask the following questions. First, do firms differentiate themselves with different prices and/or quality choices or do they make the same choices? Everyday experience suggests that price and/or quality dispersion is quite common in many markets. How to explain price dispersion was first addressed by Stigler (1961). The role of imperfect information in explaining quality dispersion has been less emphasized (but receives some attention in, e.g., Chan and Leland 1982). Second, can price act as a signal of quality to consumers who somehow cannot evaluate it? From previous literature with exogenous quality we know that the adverse selection problem can be mitigated if firms can signal quality choices to the consumers on the basis of the prices they charge (see, e.g., Bagwell and Riordan 1991). Third, how should we characterize the outcomes in terms of Pareto-efficiency?

Stigler (1961) has pointed out that acquiring information about market prices is costly. As consumers can have different search costs, different groups of consumers can be present in a market: those who know all prices and those who do not. This idea is central in Varian (1980). The idea also readily extends to quality. In Cooper and Ross (1984), for example, all consumers know prices, but some of them are informed about quality while the rest is uninformed about quality. We combine these approaches in the following way. Quality is a more complex notion than price and so it is more costly for a consumer to learn the quality than to learn the price a firm charges. We therefore assume there are three groups of consumers in our model: fully informed consumers know prices and quality of the products in a market, partially informed consumers know the prices but not the quality and fully uninformed consumers know neither prices nor quality. We emphasize the role of partially informed consumers. When they are present in a market, price is not just an instrument of competition between firms, but potentially also a signalling instrument. When they are absent, our model essentially reduces to a variation on Varian (1980) where price is replaced by a price/quality combination.

We analyse the consequences of this informational scenario in a model where two firms choose price and quality and consumers buy one good at most. Either firm is unaware of the quality choice of its competitor before it has to make its own choice over the price. The formal model is therefore one where firms choose prices and quality simultaneously. The case where firms have to choose prices while being unaware of the quality chosen by their competitors is also at the heart of Daughety and Reinganum (2005, 2007), Janssen and Roy (2010), and Janssen and van Reeven (1998). They provide examples of markets (such as markets with illegal practices and/or where safety standards are involved) where firms are indeed unaware of the quality rival firms produce. This type of markets is one motivation for our setup. The cited papers treat quality as exogenous, however, where we consider endogenous quality choice.

Another motivation for the simultaneous price/quality move stems from the observation that many firms, in consumer electronics, clothing, automobiles and other industries, compete in price/quality *bundles*: prices and quality tend to change together. For example, a new digital camera comes out with a certain price tag and usually keeps this tag during the period, in which the company itself and its competitors are quick enough to release other new cameras (hence new quality choices) along with new price tags. It is this kind of competition where quality can change as fast as price that suggests simultaneous decisions in prices and quality.

In this model with endogenous quality choice, we arrive at the following results. First, there is an equilibrium characterized by a dispersion of prices and quality where price signals quality precisely. This kind of equilibrium correspondence between price and quality is formally described as a curve in a price-quality strategy space resulting in a one-dimensional distribution of price/quality offers over that curve. In such an equilibrium partially informed consumers learn the true quality from the prices. Second, consumers' preferences over the resulting price/quality offers are monotone in price: a consumer always prefers either the cheapest offer or the most expensive one. Which of these two particular equilibria is to occur depends on how marginal utility of quality changes with respect to price. Third, though the preferences over equilibrium price/quality offers are monotone in price, equilibrium quality need not be so, e.g. the quality may be worse for average prices and better for low and high prices. Fourth, we show that price/quality combinations offered in equilibrium are Pareto-inefficient due to the signalling behaviour of firms.

The paper is organized in the following way. Section 2 formally introduces the model. Section 3 provides the equilibrium analysis. Section 4 gives an example that illustrates the more complicated expressions of section 3. Section 5 concludes. The more technical proofs of propositions are given in the appendix.

2 Model Setup

We consider a market with two firms selling similar products. The firms choose both price and quality of the product they offer. There is a unit mass of consumers who choose where to buy. The timing is as follows. First, firms simultaneously decide on the price and quality of their products. Second, each consumer decides from which firm to buy and whether to buy the product at all.

The production technology is such that producing higher quality comes at a higher cost. For simplicity we assume a linear dependency[1] so that the per-unit profits are given by

$$\Pi(p, q) = p - aq,$$

where p and q represent price and quality and the coefficient $a > 0$ characterizes the quality production technology. We take a to be the same across firms. We assume that the firms make their production costs at the moment of sale, so there are no excess goods that are produced but not sold.

Since the firms move simultaneously a strategy of either firm is simply a distribution over all possible (p, q) bundles. Let P_i and Q_i be the random variables that stand for the price and quality offered by firm i and let $\mathbb{P}_i$ be the probability measure that corresponds to the strategy of firm i.

Consumers are homogeneous in their preferences and are represented by a utility function $U(p, q)$. All consumers have the same reservation utility U_R and demand one unit of good. Total demand is normalized to 1.

[1] A more general concave function gives the same qualitative results.

Consumers search for the best price/quality combination – the one that maximizes $U(p,q)$.

As explained in the introduction, we consider three groups of consumers: (i) fully informed consumers know the prices and quality offered by both firms, (ii) partially informed consumers know the prices offered but not the quality, and (iii) fully uninformed consumers know neither price nor quality. These groups are referred to as H, M and L consumers, respectively. Their relative sizes are given by λ_H, λ_M and λ_L with $1 = \lambda_H + \lambda_M + \lambda_L$.

The H group consists of expert consumers who know the firms and the products and can check costlessly for prices and quality. A new consumer or a consumer who lacks certain expertise to assess the quality, but who can take his or her time to check for different price offers belongs to the M group. A person with substantially high alternative costs of searching for a better price and quality is a typical member of the L group.

Consumers search for the best offer given the information they have. H consumers know exactly what the offers are and know how the utility from the best offer compares against their reservation utility U_R. M and L consumers search for the best offer based on their expectations and so they do not know how the offers they choose from actually compare against their reservation utility. We assume, however, that the firms have a return policy (as is required by law in many countries), and that any consumer can learn the quality of the product he purchased within the return period.[2] Therefore, if a consumer purchases a (p,q) bundle with $U(p,q) < U_R$, the product is returned, the firm makes no profit, and the consumer gets U_R.

Let us now put the behaviour of the consumers into a more formal context. Let (p_i, q_i) be the offer of firm i. A consumer of type H knows both offers in full detail and he compares $U(p_1,q_1)$, $U(p_2,q_2)$ and U_R and chooses the option that gives the highest payoff. A consumer of type M knows only p_1 and p_2 but not q_1 or q_2. Given p_1 and p_2, he has beliefs about the distribution of Q_1 and Q_2. We use the notation $\hat{U}_j(p_i; p_{-i})$ to denote consumer j's expected utility – where the expectation is taken over his beliefs – from the offer of firm i given both the price of firm i and the price of its competitor, firm $-i$. An M consumer can buy from either firm 1 or 2, the corresponding expected payoffs are $\hat{U}_j(p_1; p_2)$ and $\hat{U}_j(p_2; p_1)$. When he buys from firm i, he learns quality q_i. Now he can decide whether to keep or to return the product. If he keeps it, he gets $U(p_i, q_i)$, otherwise he gets U_R.

Consumers of type L go to only one of the firms and buy there, and we assume that *half* of them go to either firm. Once the product is purchased, an L consumer knows the price and learns the quality, and he can either keep or return the product. He keeps it if $U(p_i, q_i) \geq U_R$.

[2] It is feasible therefore that a consumer purchases from both firms, compares the quality at home, keeps the best offer and returns the other. Such a consumer, however, values his time less and so by definition belongs to the H group.

The subsequent analysis is based on the assumption that the utility function $U(p, q)$ is well-behaved.

Assumption 1. *The utility function $U(p, q)$ is strictly decreasing in p, strictly increasing in q, strictly quasi-concave in (p, q) and twice differentiable in (p, q). Moreover, $U(p, q)$ is such that the optimization problem*

$$\max_{p,q} \Pi(p, q) \quad s.t. \quad U(p, q) \geq x$$

has a solution for any $x \geq U_R$ (in a sense, the utility function should be sufficiently quasi-concave).

2.1 Equilibria

This is a game with complete but imperfect information. So, we use the notion of sequential equilibrium (Kreps and Wilson 1982). In short, players' strategies and beliefs form a sequential equilibrium if the strategies are sequentially rational given the beliefs and the beliefs are consistent with the strategies.

In general, a certain price may signal a specific distribution of quality.[3] We restrict attention, however, to symmetric equilibria where a certain price p signals a certain quality $\hat{q}(p)$, we name function $\hat{q}(p)$ an equilibrium curve. This restriction considerably simplifies the analysis. We show that in this restricted class of equilibria interesting price and quality choices can be made. Formally, we restrict our attention to exact signalling equilibria defined as follows:

Definition 1. *A sequential equilibrium is called an exact signalling equilibrium if (i) the strategies of the firms are symmetric, i.e $\mathbb{P}_1 \equiv \mathbb{P}_2$, (ii) $supp(\mathbb{P}) = \{(p, q) : p \in [p_l, p_h], q = \hat{q}(p)\}$ and (iii) $\hat{q}(p)$ is continuously differentiable in p over $[p_l, p_h]$, where p_l and p_h are some arbitrary bounds and $\hat{q}(p)$ is some arbitrary function of p.*

It is important to note that conditions (i)-(iii) restrict the set of equilibrium strategies. We impose no restrictions on out-of-equilibrium strategies, i.e. a firm can deviate to playing any possible (p, q) bundles if it finds doing so profitable.

As the sequential rationality of the consumers' strategies is obvious from the description above, it remains to discuss the consistency of beliefs and the sequential rationality of firms' strategies.

The consumers of the H group do not hold any beliefs, because they observe prices and quality directly. The consumers of the L group, as assumed, posses trivial beliefs: half of them beliefs firm 1 has a better offer, the other half beliefs firm 2 has a better offer (these beliefs do not and can

[3] This happens if firms play some mixed strategies over a region in (p, q) space.

not depend upon the prices or quality of the offers, because L consumers do not observe any of that information). As for the consumers of the M group, their expected utility is consistent with the strategies of the firms if it is computed over the probability measure that defines those strategies. So,

$$\hat{U}_j(p_i; p_{-i}) = \mathbb{E}(\max(U(P,Q), U_R) \mid P = p_i) \quad \forall p_i \in [p_l, p_h], \tag{1}$$

where $\mathbb{E}$ is the expectation operator. We take the maximum of $U(P,Q)$ and U_R, because a consumer can choose to return the product if the realization of $U(P,Q)$ is smaller than his reservation utility U_R.

Equation (1) tells us that consistent beliefs, when considered for $p_i \in [p_l, p_h]$, result in an expected utility, which is the same for all M consumers and does not depend upon the price of the rival firm. Therefore, we use the following notation: $\hat{U}_j(p_i; p_{-i}) = \hat{U}(p_i)$.

In an exact signalling equilibrium for any realization of P there is a unique corresponding realization of $Q = \hat{q}(P)$. So, $U(P,Q) = U(P, \hat{q}(P))$ and

$$\begin{aligned}\hat{U}(p) = \mathbb{E}(\max(U(P,Q), U_R) \mid P = p) = \\ \mathbb{E}(\max(U(P, \hat{q}(P)), U_R) \mid P = p) = \max(U(p, \hat{q}(p)), U_R)\end{aligned} \tag{2}$$

for all $p \in [p_l, p_h]$. Equation (2) gives the consistency condition for beliefs. Turning to the sequential rationality of the strategies of the firms, we begin by writing down the expected profits a firm gets if it selects a particular (p,q) bundle and if its rival is playing an equilibrium strategy.

Let $\mu_H(p,q)$, $\mu_M(p)$ and μ_L denote the expected number of H type, M type and L type consumers that a firm gets if it charges (p,q) such that $U(p,q) \geq U_R$. Given the sequentially rational strategies of consumers

$$\mu_H(p,q) = \mathbb{P}(U(P,Q) < U(p,q)) \cdot \lambda_H + \mathbb{P}(U(P,Q) = U(p,q)) \cdot \frac{\lambda_H}{2}.$$

Indeed, all H consumers go to firm i if its (p,q) bundle gives higher utility than that of the rival. In general, a rival plays a mixed strategy and hence the chance to get all H consumers is given by $\mathbb{P}(U(P,Q) < U(p,q))$. If both offers give the same utility consumers split evenly.

In a similar way we have

$$\mu_M(p,q) = \mathbb{P}(\hat{U}(P) < \hat{U}(p)) \cdot \lambda_M + \mathbb{P}(\hat{U}(P) = \hat{U}(p)) \cdot \frac{\lambda_M}{2}.$$

The only difference is that M consumers do not compare actual utilities, but rather the expected utilities given the prices. For $p \in [p_l, p_h]$ the expected utility is given by equation (2), for $p \notin [p_l, p_h]$ the expected utility comes from out-of-equilibrium beliefs and, in principal, can be be arbitrary. Theorem 3, which concerns the existence of exact signalling equilibria, touches on out-of-equilibrium beliefs in more detail (see the proof of the theorem).

Finally, for the L consumers we have $\mu_L = \frac{\lambda_L}{2}$.

Define, for convenience, $\mu(p,q) = \mu_H(p,q) + \mu_M(p) + \mu_L$. Then expected profits are given by

$$\pi(p,q) = \begin{cases} \mu(p,q) \cdot \Pi(p,q) & \text{if } U(p,q) \geq U_R, \\ 0 & \text{otherwise.} \end{cases}$$

As the firms choose simultaneously, a firm's sequentially rational strategy is simply a best response strategy. Choosing a (p,q) bundle over an equilibrium curve $\hat{q}(p)$ is a best response strategy if and only if the profit function $\pi(p,q)$ attains its maximum along that equilibrium curve:

$$\text{supp}(\mathbb{P}) \in \arg\max_{p,q} \pi(p,q). \tag{3}$$

Equations (2) and (3) give necessary and sufficient conditions for there to be an exact signalling equilibrium. To have non-trivial results we make the following additional assumption:

Assumption 2. *The model is non-degenerate, i.e. there exists* (p,q) *such that* $\Pi(p,q) > 0$ *and* $U(p,q) \geq U_R$.

Since $\pi(p,q) \geq \mu_L \cdot \Pi(p,q)$ if $U(p,q) \geq U_R$, a firm can always guarantee itself some positive profits given the above assumption. So, in an equilibrium no firm will offer (p,q) such that $U(p,q) < U_R$. Consequently,

$$\hat{U}(p) = \max(U(p,\hat{q}(p)), U_R) = U(p,\hat{q}(p)) \quad \forall p \in [p_l, p_h]. \tag{4}$$

Next we rewrite μ_H and μ_M in terms of a common distribution function. Let $F(u) = \mathbb{P}(U(P,Q) < u) = \mathbb{P}(\hat{U}(P) < u)$ and $dF(u) = \mathbb{P}(U(P,Q) = u) = \mathbb{P}(\hat{U}(P) = u)$, then

$$\mu_H(p,q) = F(U(p,q)) \cdot \lambda_H + dF(U(p,q)) \cdot \frac{\lambda_H}{2},$$
$$\mu_M(p) = F(\hat{U}(p)) \cdot \lambda_M + dF(\hat{U}(p)) \cdot \frac{\lambda_M}{2}.$$

In the next section we show that in any exact signalling equilibrium $U(p,\hat{q}(p))$ is strictly monotone in p. Therefore an exact signalling equilibrium is fully characterized by its equilibrium curve $\hat{q}(p)$, by the boundary points p_l and p_h, by the distribution of utilities along the equilibrium curve, namely $F(u)$, and by its out-of-equilibrium beliefs, namely $\hat{U}(p)$ for $p \notin [p_l, p_h]$.

3 Analysis

In this section we solve for an exact signalling equilibrium, i.e. we solve for $F(u)$, $\hat{q}(p)$, p_l and p_h given $U(p,q)$ and given the other parameters of the model. At first we assume that there exists an exact signalling equilibrium and we derive its properties. Later, we also discuss existence conditions for an exact signalling equilibrium.

One of the functions that characterizes an exact signalling equilibrium is a CDF of utility over the equilibrium curve, namely $F(u)$. In this section we solve for $F(u)$.

$\hat{U}(p)$ is continuous in p because $U(p,q)$ is continuous in p and q and $\hat{q}(p)$ is differentiable. The continuity of $\hat{U}(p)$ allows us to define $[U_l, U_h] = \hat{U}([p_l, p_h])$. So, in equilibrium the corresponding utility is distributed over an interval. We next show that $F(u)$ does not have atoms.

Lemma 1. *$F(u)$ is continuous and $dF(u) \equiv 0$.*

In economic terms, the chance that the rivals provide the same utility level is zero, a result that is very similar in nature to the result that in the "model of sales" (Varian, 1980) the price distribution is atomless. The formal proof of this statement is therefore omitted. The next lemma argues that U_l must be equal to U_R. The main reason is that the firm offering the worst utility only gets uninformed consumers and if $U_l > U_R$, it could make more profit by providing them a worse deal.

Lemma 2. *$U_l = U_R$.*

Given lemmas 1, 2 and equation (4) (consistency of beliefs) it is straightforward to simplify the expression for $\pi(p,q)$.

Lemma 3. *For $p \in [p_l, p_h]$ the profits are given by*

$$\pi(p,q) = \left(F(U(p,q)) \cdot \lambda_H + F(U(p,\hat{q}(p))) \cdot \lambda_M + \frac{\lambda_L}{2} \right) \cdot \Pi(p,q)$$

if $U(p,q) \geq U_R$ and they equal 0 otherwise.

To find the functional form of $F(u)$ we need to be able to define equilibrium per-unit profits as a function of utility (lemma 5 will clarify why this is necessary). The following lemma allows us to do so.

Lemma 4. *Given $u \in [U_l, U_h]$ per-unit profits $\Pi(p, \hat{q}(p))$ are the same for all $p \in \hat{U}^{-1}(u)$.*

To understand this lemma, take an arbitrary u from $[U_l, U_h]$. The iso-utility curve corresponding to u is implicitly given by $U(p,q) = u$. This iso-utility curve will intersect the equilibrium curve $\hat{q}(p)$ at least once. If $\{(p_i, \hat{q}(p_i))\}$ is the set of intersection points, then from the definition of $\hat{U}$, $\{p_i\}$ is precisely $\hat{U}^{-1}(u)$. At each intersection point $(p_i, \hat{q}(p_i))$ we can compute the per-unit profits $\Pi(p_i, \hat{q}(p_i))$. The lemma states that $\Pi(p_i, \hat{q}(p_i))$ does not depend upon a particular choice of the intersection point, it only depends upon u. So, we use the notation $\hat{\Pi}(u)$ to denote profits a firms obtains by offering a utility level $u(p,q)$.

Formally, take an arbitrary $\tilde{p}(u)$ such that $\tilde{p}(u) \in \hat{U}^{-1}(u)$ for all $u \in [U_l, U_h]$. Then

$$\hat{\Pi}(u) = \Pi(\tilde{p}(u), \hat{q}(\tilde{p}(u))). \tag{5}$$

It is not possible to define $\hat{\Pi}(u)$ explicitly as it involves choosing a particular $\tilde{p}(u)$ and the functional form of $U(p,q)$ is not given. However, once a specific functional form of $U(p,q)$ is adopted, and once $\hat{q}(p)$ is known, it is possible to choose a particular $\tilde{p}(u)$ and hence solve for $\hat{\Pi}(u)$.

Now we can solve for the functional form of $F(u)$ using techniques that are known in the search literature.

Lemma 5.

$$F(u) = \frac{1}{2} \cdot \frac{\lambda_L}{\lambda_H + \lambda_M} \left(\frac{\hat{\Pi}(U_R)}{\hat{\Pi}(u)} - 1 \right) \quad \textit{for} \quad u \in [U_l, U_h].$$

Proof. It follows from lemma 3 that

$$\pi(p, \hat{q}(p)) = \left(F(U(p, \hat{q}(p))) \cdot (\lambda_H + \lambda_M) + \frac{\lambda_L}{2} \right) \cdot \Pi(p, \hat{q}(p)). \tag{6}$$

Evaluating (6) at $\tilde{p}(u)$, noticing that $U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u$ and using the definition of $\hat{\Pi}(u)$ gives

$$\pi(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = \left(F(u) \cdot (\lambda_H + \lambda_M) + \frac{\lambda_L}{2} \right) \cdot \hat{\Pi}(u).$$

For there to be an equilibrium the strategies of the firms should be sequentially rational. Therefore $\pi(p, \hat{q}(p))$ is constant over this interval, and we denote its value by $\hat{\pi}$. Since $\tilde{p}(u) \in [p_l, p_h]$ we get

$$\hat{\pi} = \left(F(u) \cdot (\lambda_H + \lambda_M) + \frac{\lambda_L}{2} \right) \cdot \hat{\Pi}(u).$$

By definition, $F(U_l) = 0$. Also, $U_l = U_R$. So,

$$\hat{\pi} = \frac{\lambda_L}{2} \cdot \hat{\Pi}(U_R).$$

Plugging it back and solving for $F(u)$ gives the result. □

It then follows that U_h is implicitly given by $F(U_h) = 1$. Using lemma 5 it may alternatively be given by

$$\hat{\Pi}(U_h) = \frac{1/2 \cdot \lambda_L \cdot \hat{\Pi}(U_R)}{\lambda_H + \lambda_M + 1/2 \cdot \lambda_L}.$$

The previous lemma shows that the distribution of utility levels over the equilibrium curve is such that the fraction of uninformed consumers to the other consumers determines the spread of the utility. This is intuitive as the firms have market power over these uninformed consumers and if there are many of them, the price quality offers concentrate around the offers that are cheapest to provide. We next provide a description of the equilibrium curve $\hat{q}(p)$.

Lemma 6. *If there is an exact signalling equilibrium then $\hat{q}(p)$ has to satisfy*

$$\frac{d\hat{q}}{dp} = -\frac{\lambda_H + \lambda_M}{\lambda_M} \cdot \frac{U_p'(p, \hat{q}(p))}{U_q'(p, \hat{q}(p))} - \frac{\lambda_H}{a\,\lambda_M}$$

everywhere on (p_l, p_h).

Proof. It should be that

$$\frac{\partial \pi(p,q)}{\partial p}\Big|_{(\tilde{p},\hat{q}(\tilde{p}))} = 0, \qquad \frac{\partial \pi(p,q)}{\partial q}\Big|_{(\tilde{p},\hat{q}(\tilde{p}))} = 0,$$

because otherwise a firm may get higher profits by deviating along the gradient vector. Using lemma 3 we get

$$\begin{aligned}
\frac{\partial \pi(p,q)}{\partial p}\Big|_{(\tilde{p},\hat{q}(\tilde{p}))} &= \Big(F'(U(p,q)) \cdot U_p'(p,q) \cdot \lambda_H + \\
&F'(U(p,\hat{q}(p))) \cdot \big(U_p'(p,\hat{q}(p)) + U_q'(p,\hat{q}(p)) \cdot \hat{q}'(p)\big) \cdot \lambda_M\Big) \cdot (p - aq) + \\
&\Big(F(U(p,q)) \cdot \lambda_H + F(U(p,\hat{q}(p))) \cdot \lambda_M + \frac{\lambda_L}{2}\Big) \cdot 1\Big|_{(\tilde{p},\hat{q}(\tilde{p}))} = \\
&F'(U(\tilde{p},\hat{q}(\tilde{p})))\Big(U_p'(\tilde{p},\hat{q}(\tilde{p}))(\lambda_H + \lambda_M) + U_q'(\tilde{p},\hat{q}(\tilde{p})) \cdot \hat{q}'(\tilde{p}) \cdot \lambda_M\Big) \cdot \\
&(\tilde{p} - a\hat{q}(\tilde{p})) + \Big(F(U(\tilde{p},\hat{q}(\tilde{p})) \cdot (\lambda_H + \lambda_M) + \frac{\lambda_L}{2}\Big) = 0 \qquad (7)
\end{aligned}$$

and

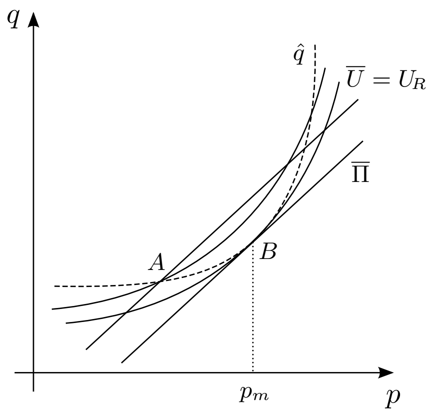
$$\begin{aligned}
\frac{\partial \pi(p,q)}{\partial q}\Big|_{(\tilde{p},\hat{q}(\tilde{p}))} &= \Big(F'(U(p,q)) \cdot U_q'(p,q) \cdot \lambda_H\Big) \cdot (p - aq) + \\
&\Big(F(U(p,q)) \cdot \lambda_H + F(U(p,\hat{q}(p))) \cdot \lambda_M + \frac{\lambda_L}{2}\Big) \cdot (-a)\Big|_{(\tilde{p},\hat{q}(\tilde{p}))} = \\
&\Big(F'(U(\tilde{p},\hat{q}(\tilde{p}))) \cdot U_q'(\tilde{p},\hat{q}(\tilde{p})) \cdot \lambda_H\Big) \cdot (\tilde{p} - a\hat{q}(\tilde{p})) + \\
&\Big(F(U(\tilde{p},\hat{q}(\tilde{p})) \cdot (\lambda_H + \lambda_M) + \frac{\lambda_L}{2}\Big) \cdot (-a) = 0. \qquad (8)
\end{aligned}$$

From (7) and (8) it follows after some algebra that

$$\frac{d\hat{q}}{d\tilde{p}} = -\frac{\lambda_H + \lambda_M}{\lambda_M} \cdot \frac{U_p'(\tilde{p}, \hat{q}(\tilde{p}))}{U_q'(\tilde{p}, \hat{q}(\tilde{p}))} - \frac{\lambda_H}{a\,\lambda_M}.$$

□

Figure 1: Equilibrium Curve



Notation: $\overline{X}$ stands for $X(p,q) = \text{const}$, where constant is arbitrary; $\overline{X} = X_0$ stands for $X(p,q) = X_0$, where X_0 is some specific value.

To illustrate the impact of the lemma, figure 1 depicts an equilibrium curve $\hat{q}(p)$ together with iso-utility curves and isolines of per-unit profits.

To see why an equilibrium curve has a shape as given in the figure, rewrite the differential equation for $\hat{q}(p)$ as follows:

$$\frac{d\hat{q}}{dp} = \frac{\lambda_H}{\lambda_M}\left(-\frac{U'_p(p,\hat{q}(p))}{U'_q(p,\hat{q}(p))} - \frac{1}{a}\right) - \frac{U'_p(p,\hat{q}(p))}{U'_q(p,\hat{q}(p))}, \tag{9}$$

and recall that the isoline of per-unit profits has a slope equal to $1/a$. The slope of the iso-utility curves is $-\frac{U'_p(p,q)}{U'_q(p,q)}$. Therefore it follows from (9) that if the slope of an iso-utility curve is less than $1/a$ (point A, for example), the slope of an equilibrium curve is even smaller at that point and vice versa. If the slope is exactly $1/a$ (point B), then an iso-utility curve and an equilibrium curve are tangent to each other and they are also tangent to an isoline of per-unit profits at that point. Therefore an equilibrium curve relative to iso-utility curves should look as depicted in the figure.

According to lemma 2 the lowest attainable utility along an equilibrium curve equals U_R , thus the equilibrium curve in the figure "lies" on an iso-utility curve that corresponds to $U = U_R$.

To find out the boundary points of an equilibrium curve, let us refer to figure 1 once more. An equilibrium curve spans $[p_l, p_h]$, by definition. The figure shows that a choice of $[p_l, p_h]$ has important economic consequences: if $[p_l, p_h]$ is located to the left of the point of tangency p_m then $U(p, \hat{q}(p))$ is decreasing in p, i.e. *lower* prices signal higher utility. If, on the contrary, $[p_l, p_h]$ is to the right of p_m then $U(p, \hat{q}(p))$ is increasing in p and *higher* prices signal higher utility.

We find that both cases are possible depending upon the properties of $U(p,q)$, but that in both cases p_m is the boundary point of the price interval. To state the main result, we need two definitions. First, we formally define point (p_m, q_m):

Definition 2. *The point (p_m, q_m) is uniquely*[4] *defined by*

$$(p_m, q_m) = \arg\max_{(p,q):U(p,q)\geq U_R} \Pi(p,q).$$

Lemma A.1 (appendix) shows that the point (p_m, q_m) defined in this way belongs to the equilibrium curve $\hat{q}(p)$ and that the equilibrium utility $U(p, \hat{q}(p))$ attains its minimum at p_m – just like it is in the figure.

Second, we define a contract curve in the usual way as a curve that consists of all Pareto-efficient allocations in (p,q) plane:

Definition 3. *Let*

$$(p^*(x), q^*(x)) = \arg\max_{(p,q):U(p,q)\geq x} \Pi(p,q).$$

Then, if there exists a function g such that $q^*(x) = g(p^*(x))$, we shall refer to this function as a contract curve.

As, in principle, it is not necessary that $g(p)$ is defined for every p, we make the following technical assumption:

Assumption 3. *A contract curve $g(p)$ is defined in the neighbourhood of p_m and is differentiable at this point.*

Now we can state the following theorem:

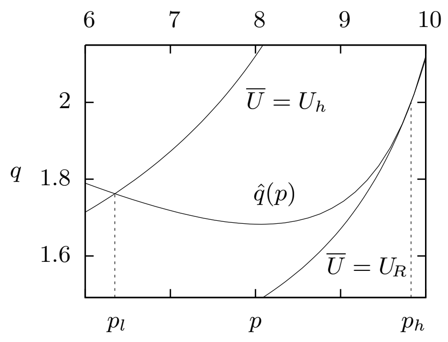
Theorem 1. *If $g'(p_m) < \frac{1}{a}$ and if there exists an exact signalling equilibrium then $[p_l, p_h] = [p_l, p_m]$ and $U(p, \hat{q}(p))$ is strictly decreasing in p over this interval. Hence in such an equilibrium higher prices signal lower utility.*

If $g'(p_m) > \frac{1}{a}$ and if there exists an exact signalling equilibrium then $[p_l, p_h] = [p_m, p_h]$ and $U(p, \hat{q}(p))$ is strictly increasing in p over this interval. Hence in such an equilibrium higher prices signal higher utility.

Theorem 1 explains that the signalling equilibrium relation between price and utility is monotone and can be of two different types. The main intuition for this result can be explained by reference to lemma 3 and figure 1. The profit of a firm along the equilibrium curve should be constant. Both the fully and partially informed consumers buy at the firm where the perceived utility is highest. Below the point p_m in figure 1, this implies that these consumers buy at the lowest possible price. To make that firms are indifferent between any point on the equilibrium curve, lemma 3 tells us that

[4] The solution to the optimization problem exists by assumption 1. Moreover, the solution is unique because $U(p,q)$ is strictly quasi-concave by the same assumption.

Figure 2: Non-monotonic $\hat{q}(p)$



Notation: $\overline{U} = u$ stands for $U(p,q) = u$.

on that part of the equilibrium curve lower prices should be accompanied by lower per-unit profits and this is accomplished by a slope of the equilibrium curve strictly below $1/a$. The reverse argument holds true for points on the equilibrium curve above p_m. However, the theorem says nothing about the monotonicity of the equilibrium quality, which is $\hat{q}(p)$. It turns out that $\hat{q}(p)$ is not necessarily monotone. Figure 2 illustrates a particular exact signalling equilibrium with a non-monotone curve.[5]

If the marginal utility of quality declines as the price increases, i.e., if $U_{qp} < 0$, then $g(p)$ has a negative slope and $[p_l, p_h] = [p_l, p_m]$, so an equilibrium where *lower* prices signal higher utility results.

Given Theorem 1 we can solve for the boundary points p_l and p_h. If $g'(p_m) < \frac{1}{a}$ then $p_h = p_m$. As in this case $\hat{U}(p) = U(p, \hat{q}(p))$ is strictly decreasing in p, $p_l = \hat{U}^{-1}(U_h)$.Similarly, if $g'(p_m) > \frac{1}{a}$, $p_l = p_m$ and $p_h = \hat{U}^{-1}(U_h)$.

3.1 Existence and Uniqueness

In the previous sections we have uniquely determined all the parameters of an exact signalling equilibrium (except for out-of-equilibrium beliefs). Therefore we have the following theorem:

Theorem 2. *There is at most one exact signalling equilibrium(up to out-of-equilibrium beliefs).*

Proof. We recollect some of the results obtained so far. (p_m, q_m) was uniquely defined in def. 2. $\hat{q}(p)$ is given by a differential equation of lemma 6 and it is

[5]We used the following example to build the figure: $U(p,q) = (q-1)^{1/2}(10-p)^{1/2}$, $\lambda_H = 0.05$, $\lambda_M = 0.1$, $\lambda_L = 0.85$, $U_R = 1$ and $a = 1$. Section 4 gives another example and shows how to solve for an exact signalling equilibrium. If to apply that procedure to this example, one will get precisely fig. 2. However, this particular example we do not discuss in detail as the computations are harder comparing with the example of section 4.

known to go through (p_m, q_m) (lemma A.1), (p_m, q_m) is thus the boundary point to uniquely solve the differential equation. Depending upon the sign of $g'(p_m)$ we know that an exact signalling equilibrium spans either $[p_l, p_m]$ or $[p_m, p_h]$. In either case $U(p, \hat{q}(p))$ is strictly monotone (theorem 1) and therefore $\tilde{p}(u)$ is uniquely determined by $U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u$. In turn, $\tilde{p}(u)$ gives us $\hat{\Pi}(u)$ and $\hat{\Pi}(u)$ gives $F(u)$ (see eq. (5) and lemma 5 respectively). Equation $F(U_h) = 1$ uniquely determines U_h and from U_h we can determine the remaining p_l (or p_h). Hence we can uniquely determine the parameters of an exact signalling equilibrium, namely $\hat{q}(p)$, p_l, p_h and $F(u)$. □

Next, we address the existence issue. The main issue here is the following. Note that for any given utility function $U(p, q)$ and given the rest of the parameters (λ_H, λ_M, λ_L and a) we can always find an equilibrium curve $\hat{q}(p)$, its boundary points p_l and p_h and the distribution of utility over that curve, namely $F(u)$. We also know that profit function $\pi(p, q)$ will be constant along $\hat{q}(p)$ as required. But none of the results obtained so far guarantees that $\pi(p, q)$ will attain its maximum over $\hat{q}(p)$. Unfortunately, for a general utility specification it is impossible to provide sufficient existence conditions that do not involve a complete solution of the model. The difficulty that arises is that there is no explicit expression for the profit function at points that are off the equilibrium curve.[6] However, we can address the question of existence from a different angle: given an arbitrary equilibrium curve $\hat{q}(p)$, can we find such parameters of our model that there is a corresponding exact signalling equilibrium, i.e. one that has $\hat{q}(p)$ as its equilibrium curve? The following theorem 3 provides the answer, but first we need to formulate two additional necessary conditions.

Consider an arbitrary strictly increasing, strictly convex and twice differentiable equilibrium curve $\hat{q}(p)$ defined over some $[p_l, p_h]$. If we are looking for a corresponding equilibrium where lower prices signal higher utility, then $p_m = p_h$ and, consequently, $\hat{q}'(p_h) = \frac{1}{a}$. Hence, we know a and we can evaluate per-unit profits $\Pi(p, q)$. If there is an equilibrium it should be that

$$\Pi(p, \hat{q}(p)) = p - a\hat{q}(p) = p - \frac{\hat{q}(p)}{\hat{q}'(p_h)} > 0 \tag{10}$$

for all $p \in [p_l, p_h]$.

For a strictly increasing and strictly convex $\hat{q}(p)$ per-unit equilibrium profits $\Pi(p, \hat{q}(p))$ are strictly increasing in p over $[p_l, p_m]$ and therefore (10) is equivalent to

$$\Pi(p_l, \hat{q}(p_l)) = p_l - \frac{\hat{q}(p_l)}{\hat{q}'(p_h)} > 0. \tag{11}$$

[6] In a note for interested readers we provide an existence theorem for a specific linear-kinked utility function.

In a similar way, if we are looking for a corresponding equilibrium where higher prices signal higher utility, it should be that

$$\Pi(p_h, \hat{q}(p_h)) = p_h - \frac{\hat{q}(p_h)}{\hat{q}'(p_l)} > 0. \tag{12}$$

Given these two conditions we can state the theorem.

Theorem 3. *Consider an arbitrary strictly increasing, strictly convex and twice differentiable equilibrium curve $\hat{q}(p)$ defined over $[p_l, p_h]$ and satisfying* (11) *or* (12) *or both. Then there exist a utility function $U(p, q)$ satisfying assumption 1, parameters $(U_R, \lambda_H, \lambda_M, \lambda_L, a)$ and out-of-equilibrium beliefs such that there will be a corresponding exact signalling equilibrium, i.e one that has $\hat{q}(p)$ as its equilibrium curve.*

In other words, it may not be possible to have an exact signalling equilibrium for any $U(p, q)$, but at least there will be exact signalling equilibria for as many different forms of $U(p, q)$ as to generate every possible strictly increasing, strictly convex equilibrium curve $\hat{q}(p)$ that allows for positive per-unit profits.

3.2 Pareto-efficiency

An allocation is Pareto-efficient in this model if an iso-utility curve is tangent to an isoline of per-unit profits. Considering figure 1, one can see that for any $(p, \hat{q}(p))$ with $p \in [p_l, p_h)$ this is not the case. Therefore, equilibrium allocations are almost surely Pareto-inefficient.

This result may not be surprising as such, but it marks a sharp difference with Varian's model of sales, which is essentially this model (with prices being replaced by utilities) when there are no partially-informed consumers. In that model, all the equilibrium allocations will be Pareto-efficient. The presence of partially-informed consumers and the incentives they create for firms to signal quality with price is what brings Pareto-inefficiency. Fully uninformed consumers do not create Pareto-inefficiency on their own, they merely create a redistribution in welfare.

4 An Example

In this section, we illustrate an exact signalling equilibrium with an example. Take

$$U(p, q) = \frac{1}{2} \ln q - p,\ U_R = -2,\ \lambda_H = \lambda_M = \frac{1}{5},\ \lambda_L = \frac{3}{5},\ a = 1.$$

We begin by solving for (p_m, q_m). To do so we solve

$$\max_{p,q} \Pi(p, q) \quad \text{s.t.} \quad U(p, q) \geq U_R \tag{13}$$

and obtain

$$p_m = \frac{1}{2}\ln\frac{1}{2} - U_R = \frac{1}{2}\ln\frac{1}{2} + 2, \quad q_m = \frac{1}{2}.$$

Next we shall check whether it's an equilibrium where higher prices signal lower utility or the one where higher prices signal higher utility. To do it we need to know $g'(p_m)$. From (13) one can readily see that q_m does not depend upon U_R and hence contract curve $g(p) = q_m = \frac{1}{2}$. Therefore $g'(p_m) = 0 < \frac{1}{a} = 1$ and we have to search for an equilibrium to the left of p_m, i.e. $[p_l, p_h] = [p_l, p_m]$ (see theorem 1).

Let us now find $\hat{q}(p)$. Plugging our utility and the parameters into the differential equation for $\hat{q}(p)$ (see lemma 6) gives

$$\frac{d\hat{q}(p)}{dp} = 4\hat{q}(p) - 1.$$

Solving it and using the boundary condition $\hat{q}(p_m) = q_m$ gives:

$$\hat{q}(p) = e^{4p-8} + \frac{1}{4}.$$

To find utility distribution $F(u)$ we need to know $\hat{\Pi}(u)$ and for that we need to find $\tilde{p}(u)$ such that $U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u$. Writing down this latter expression gives

$$\frac{1}{2}\ln\left(e^{4\tilde{p}(u)-8} + \frac{1}{4}\right) - \tilde{p}(u) = u.$$

A little bit of algebra gives the solution:

$$\tilde{p}(u) = \frac{1}{2}\ln\left(\frac{1}{2}\left(e^{2u} - \sqrt{e^{4u} - e^{-8}}\right)\right) + 4. \tag{14}$$

Having (4) and (14) we therefore also have

$$\hat{\Pi}(u) = \tilde{p}(u) - a\hat{q}(\tilde{p}(u)) = \tilde{p}(u) - \hat{q}(\tilde{p}(u))$$

and

$$F(u) = \frac{1}{2}\cdot\frac{\lambda_L}{\lambda_H + \lambda_M}\left(\frac{\hat{\Pi}(U_R)}{\hat{\Pi}(u)} - 1\right) = \frac{3}{4}\left(\frac{\hat{\Pi}(U_R)}{\hat{\Pi}(u)} - 1\right).$$

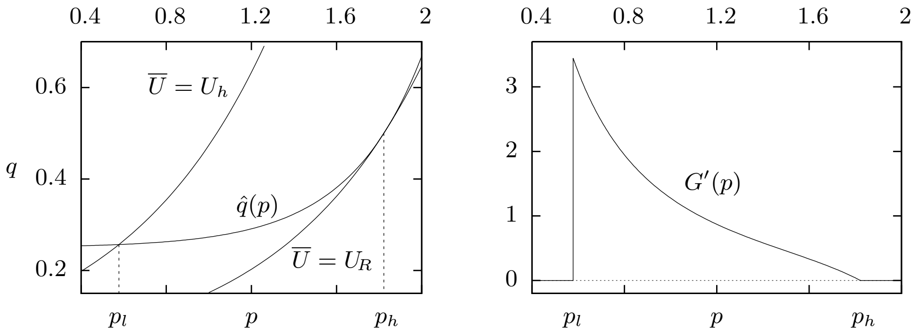
Given $F(u)$ we can find U_h from $F(U_h) = 1$. Define

$$z = \frac{1}{2}\left(e^{2U_h+4} - \sqrt{e^{4U_h+8} - 1}\right).$$

This way $z \leq \frac{1}{2}$ and $F(U_h) = 1$ can be rewritten as

$$\frac{1}{2}\ln z - z^2 + \frac{7}{4} - \frac{3}{7}\hat{\Pi}(U_R) = 0.$$

Figure 3: Equilibrium Characteristic Functions



Notation: $\overline{U} = u$ stands for $U(p, q) = u$.

This equation can not be solved analytically but a numerical solution is easily obtainable: $z \approx 0.08226$. Then, from the definition of z,

$$U_h = \frac{1}{2} \ln \left(\frac{4z^2 + 1}{4z} \right) - 2 \approx -1.43089.$$

Finally, $p_l = \tilde{p}(U_h) \approx 0.75109$.

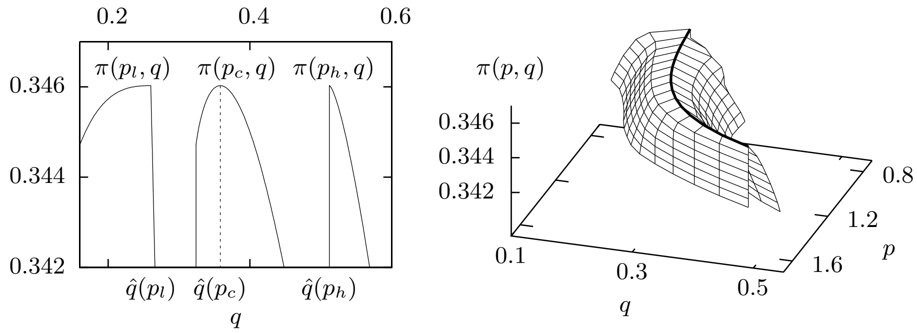
Figure 3 plots a few important functions of our equilibrium candidate. The left plot gives $\hat{q}(p)$ together with iso-utility curves that correspond to $U_l = U_R$ and U_h. It is easy to see in the plot that higher prices signal lower utility in an equilibrium. In other words, if a partially informed consumer faces two products with different prices he will go for the cheapest product and, though the expected quality will be lower, the expected utility will be higher. The right plot gives the density function of the price distribution. Earlier we exclusively worked with utility distribution $F(u)$, but there is an easy transformation as

$$G(p) = \mathbb{P}(P < p) = \mathbb{P}(U(P, \hat{q}(P)) > U(p, \hat{q}(p))) = \\ 1 - \mathbb{P}(U(P, \hat{q}(P)) \leq U(p, \hat{q}(p))) = 1 - F(U(p, \hat{q}(p))).$$

The right plot gives the density of this distribution, namely $G'(p)$. From it we can see that the lower prices, lower quality occur more often than the higher prices, higher quality.

Recollect that $\pi(p, q)$ gives expected profits of one firm when the other firm is playing the equilibrium strategy. For there to be an equilibrium it should be that $\pi(p, q)$ attains its maximum along the equilibrium curve $\hat{q}(p)$. Figure 4 plots $\pi(p, q)$. The left plot gives 2D slices of $\pi(p, q)$ for various p, the right plot attempts a 3D presentation. One can readily see that the condition in question is satisfied indeed and so we have an exact signalling equilibrium.

Figure 4: Equilibrium Profits



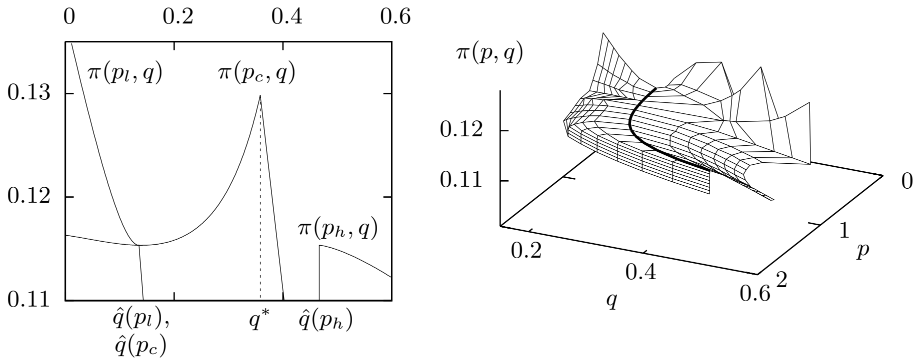
Comments: $p_c = \frac{1}{4}p_l + \frac{3}{4}p_h$ (the left plot); the bold line depicts $\pi(p, \hat{q}(p))$, i.e. the profits along the equilibrium curve (the right plot); also, for convenience, only a summit of $\pi(p, q)$ is shown in the right plot.

Does an exact signalling equilibrium always exist? Not necessary. Consider the same example but with $\lambda_L = \frac{1}{5}$ and $\lambda_H = \lambda_M = \frac{2}{5}$. It can be solved in the same way as before. Figure 5 gives the same plots as before but for this new example. We know that if there was an equilibrium it should have had the same $\pi(p, q)$ as we have found, but we have found $\pi(p, q)$ that does not have its maximum along the equilibrium curve. Hence we can conclude that there is no exact signalling equilibrium in this latter case.

5 Conclusions

We have considered a market where oligopolistic firms compete for consumers by varying prices and quality of their products and where consumers are heterogeneous in their knowledge of the prices and quality of the products offered: some know both the quality and prices, some know only the prices and some know neither. We have derived a signalling equilibrium for this setting that is characterized by firms playing a mixed strategy over a curve in a price-quality space. We have shown that this signalling equilibrium can be of two types. Both types are characterized by a dispersion of prices and quality and by Pareto-inefficiency of the price/quality offers. But in one type of equilibrium *lower* prices signal better price/quality ratios, while in the other type *higher* prices signal better price/quality ratios. Which type results depends on consumers' preferences: the cheapest offer is the best deal from a consumer perspective if the marginal utility of quality is declining in prices.

Figure 5: Disequilibrium Profits



Comments: $p_c = \frac{4}{5}p_l + \frac{1}{5}p_h$ and q^* is such that $U(p_c, q^*) = U_h$ (the left plot); the bold line depicts $\pi(p, \hat{q}(p))$, i.e. the profits along the equilibrium curve (the right plot); also, for convenience, only a summit of $\pi(p, q)$ is shown in the right plot; the sharp spokes to the back and right of the 3D picture are rendering artifacts – should a one steep "hill".

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Appendix

Lemma 2. *$U_l = U_R$.*

Proof. Consider $p \in [p_l, p_h]$ such that $U(p, \hat{q}(p)) = U_l$. Such p should exist because U_l belongs to the support of $F(u)$ by definition. Also, by definition, $F(U_l) = 0$. Therefore

$$\pi(p, \hat{q}(p)) = \frac{\lambda_L}{2} \cdot \Pi(p, \hat{q}(p)).$$

Clearly, $U_l \geq U_R$. Suppose that $U_l = U(p, \hat{q}(p)) > U_R$. Since $\Pi(p, q)$ is strictly decreasing in q and $U(p, q)$ is continuous in q, it is possible to choose such $\varepsilon > 0$ that

$$U(p, \hat{q}(p) - \varepsilon) > U_R \quad \text{and} \quad \Pi(p, \hat{q}(p) - \varepsilon) > \Pi(p, \hat{q}(p)).$$

Also, $F(U(p, \hat{q}(p) - \varepsilon)) = 0$ and therefore

$$\pi(p, \hat{q}(p) - \varepsilon) = \frac{\lambda_L}{2} \cdot \Pi(p, \hat{q}(p) - \varepsilon) > \frac{\lambda_L}{2} \cdot \Pi(p, \hat{q}(p)) = \pi(p, \hat{q}(p)).$$

This contradicts

$$(p, \hat{q}(p)) \in \arg\max_{(\tilde{p}, \tilde{q})} \pi(\tilde{p}, \tilde{q}).$$

So, $U_l = U_R$. □

Lemma 4. *Given $u \in [U_l, U_h]$ per-unit profits $\Pi(p, \hat{q}(p))$ are the same for all $p \in \hat{U}^{-1}(u)$.*

Proof. Take $p \in [p_l, p_h]$. It follows from lemma 3 that

$$\pi(p, \hat{q}(p)) = \left(F(\hat{U}(p)) \cdot (\lambda_H + \lambda_M) + \frac{\lambda_L}{2} \right) \cdot \Pi(p, \hat{q}(p)). \tag{15}$$

If there are different p_1, p_2 such that

$$\hat{U}(p_1) = \hat{U}(p_2) = u,$$

then $\Pi(p_1, \hat{q}(p_1)) = \Pi(p_2, \hat{q}(p_2))$. Indeed, if this is not the case, then equilibrium profits, i.e. the profits along an equilibrium curve, will differ between p_1 and p_2 as readily seen from (15). But profits have to attain their maximum along the equilibrium curve and hence they have to be constant along it as well. □

Theorem 1. *If $g'(p_m) < \frac{1}{a}$ and if there exists an exact signalling equilibrium then $[p_l, p_h] = [p_l, p_m]$ and $U(p, \hat{q}(p))$ is strictly decreasing in p over this interval. Hence in an equilibrium higher prices signal lower utility.*

And vice versa. If $g'(p_m) > \frac{1}{a}$ and if there exists an exact signalling equilibrium then $[p_l, p_h] = [p_m, p_h]$ and $U(p, \hat{q}(p))$ is strictly increasing in p over this interval. Hence in an equilibrium higher prices signal higher utility.

Proof. The formal proof is fully contained in the following lemmas A.1-A.5. Next we only give a bit of explanation. For there to be an equilibrium, the profit function $\pi(p, q)$ should attain its maximum along the equilibrium curve $\hat{q}(p)$ or otherwise the firms will deviate from playing (p, q) bundles over it. The idea of the proof is to apply second order necessary conditions, i.e. the conditions for local concavity, to check whether $\pi(p, q)$ can indeed attain its maximum over $\hat{q}(p)$ given different choices of p_l and p_h. In general we have to consider a Hessian to check that but for this proof it suffices and it is convenient to check concavity only in q, i.e. we look at the following second order necessary condition:

$$\left.\frac{\partial^2 \pi(p, q)}{\partial^2 q}\right|_{(p, \hat{q}(p))} \leq 0 \quad \text{for} \quad p \in [p_l, p_h].$$

Lemma A.3 provides us with $\left.\frac{\partial^2 \pi(p,q)}{\partial^2 q}\right|_{(p,\hat{q}(p))}$. However, the expression is complicated and it is hard to evaluate its sign for an arbitrary p from $[p_l, p_h]$, but important conclusions can be made when considering a limiting case with $p \to p_m$. Lemma A.4 considers the limiting case and concludes that either $[p_l, p_h] = [p_l, p_m]$ or $[p_l, p_h] = [p_m, p_h]$. Which of these cases occurs depends upon the sign of $U_{pq}(p_m, \hat{q}(p_m)) + \frac{1}{a} U_{qq}(p_m, \hat{q}(p_m))$. This latter expression does not have an immediate interpretation but it can be rewritten so as to allow for an economic one. Namely, this expression can be formulated in terms of a slope of a contract curve at p_m, which is $g'(p_m)$. Lemma A.5 does so. Together with lemma A.4 they give: if $g'(p_m) < \frac{1}{a}$ then $[p_l, p_h] = [p_l, p_m]$ and if $g'(p_m) > \frac{1}{a}$ then $[p_l, p_h] = [p_m, p_h]$. Finally, lemma A.2 gives that $U(p, \hat{q}(p))$ is strictly decreasing in p for $p < p_m$ and is strictly increasing in p for $p > p_m$. □

Lemma A.1. *$q_m = \hat{q}(p_m)$ and $p_m \in [p_l, p_h]$, i.e. point (p_m, q_m) belongs to an equilibrium curve. Moreover, $U(p, \hat{q}(p)) > U(p_m, \hat{q}(p_m))$ for all $p \neq p_m$, i.e. point (p_m, q_m) gives the minimum utility among all the points of an equilibrium curve.*

Proof. Let $A = \{p \in [p_l, p_h] \,|\, U(p, \hat{q}(p)) = U_R\}$. A denotes the prices that, together with their equilibrium qualities, provide the lowest possible utility. Set A is nonempty as follows from the definition of U_l and from the result that $U_l = U_R$ (see lemma 2). Pick an arbitrary $p_0 \in A$. Let $q_0 = \hat{q}(p_0)$. Since $U(p_0, q_0) = U_R$ we have that $F(U(p_0, q_0)) = 0$ and therefore

$$\pi(p_0, q_0) = \frac{\lambda_L}{2}\Pi(p_0, q_0).$$

As (p_0, q_0) belongs to the equilibrium curve, it maximizes the profits. Hence $\pi(p_0, q_0) \geq \pi(p, q)$ for any (p, q). Trivially, $\pi(p, q) \geq \frac{\lambda_L}{2}\Pi(p, q)$ if $U(p, q) \geq U_R$, therefore

$$\frac{\lambda_L}{2}\Pi(p_0, q_0) \geq \frac{\lambda_L}{2}\Pi(p, q)$$

for any (p, q) such that $U(p, q) \geq U_R$. Hence, (p_0, q_0) is a solution to the following optimization problem:

$$\max_{p,q} \Pi(p, q) \quad \text{s.t.} \quad U(p, q) \geq U_R.$$

We already know that this optimization problem has a unique solution, which is denoted by (p_m, q_m). So, $(p_0, q_0) = (p_m, q_m)$. To prove the second proposition it suffices to notice that since (p_0, q_0) is uniquely defined, set A consist of a single point. $\square$

Lemma A.2. *$\frac{d}{dp}U(p, \hat{q}(p)) > 0$ for $p > p_m$ and $\frac{d}{dp}U(p, \hat{q}(p)) < 0$ for $p < p_m$.*

Proof. For convenience let $\hat{U}$ stand for $U(p, \hat{q}(p))$ and let the same be for the derivatives, e.g. $\hat{U}_p$ stands for $U_p(p, \hat{q}(p)) = \frac{\partial U(p,q)}{\partial p}\Big|_{(p,\hat{q}(p))}$. Suppose $\frac{d\hat{U}}{dp} < 0$ at some point $p_0 > p_m$. Utility function $U(p, q)$ and equilibrium curve $\hat{q}(p)$ are continuous by assumption, therefore $U(p, \hat{q}(p))$ is continuous. Also $U(p_0, \hat{q}(p_0)) > U(p_m, \hat{q}(p_m))$ by lemma A.1 and $U(p, \hat{q}(p))$ is decreasing at point p_0 by the above supposition. Since $p_0 > p_m$ we can then find $p_1 \in (p_m, p_0)$ such that

$$U(p_1, \hat{q}(p_1)) = U(p_0, \hat{q}(p_0)) \quad \text{and} \quad \frac{d\hat{U}}{dp}\Big|_{p_1} > 0.$$

Let us expand $\frac{d\hat{U}}{dp}$:

$$\frac{d}{dp}U(p, \hat{q}(p)) = \hat{U}_p + \hat{U}_q \frac{d\hat{q}}{dp} = \frac{\lambda_H \hat{U}_q}{\lambda_M}\left(-\frac{\hat{U}_p}{\hat{U}_q} - \frac{1}{a}\right), \tag{16}$$

where the expression for $\hat{q}'(p)$ comes from lemma 6. Using the above to rewrite $\frac{d\hat{U}}{dp}\Big|_{p_0} < 0$ and $\frac{d\hat{U}}{dp}\Big|_{p_1} > 0$ gives

$$-\frac{U_p(p_0, \hat{q}(p_0))}{U_q(p_0, \hat{q}(p_0))} < \frac{1}{a} \quad \text{and} \quad -\frac{U_p(p_1, \hat{q}(p_1))}{U_q(p_1, \hat{q}(p_1))} > \frac{1}{a}. \tag{17}$$

Let us now consider an iso-utility curve that goes through $(p_0, \hat{q}(p_0))$ and $(p_1, \hat{q}(p_1))$. It's the same iso-utility curve because $U(p_0, \hat{q}(p_0)) = U(p_1, \hat{q}(p_1))$. Denote this curve by $\tilde{q}(p)$, i.e. $\tilde{q}(p)$ is implicitly defined by

$$U(p, \tilde{q}(p)) = U(p_1, \hat{q}(p_1)) = U(p_0, \hat{q}(p_0)). \tag{18}$$

This definition is valid since $U(p,q)$ is strictly increasing in q and so there is only one solution for $\tilde{q}$ in the above equation. For the same reason

$$\hat{q}(p_0) = \tilde{q}(p_0) \quad \text{and} \quad \hat{q}(p_1) = \tilde{q}(p_1). \tag{19}$$

Differentiating (18) gives

$$\frac{d\tilde{q}}{dp} = -\frac{U_p(p,\tilde{q}(p))}{U_q(p,\tilde{q}(p))}. \tag{20}$$

Bringing together (17), (19) and (20) gives

$$\tilde{q}'(p_0) < \frac{1}{a} < \tilde{q}'(p_1). \tag{21}$$

At the same time $U(p,q)$ is strictly decreasing in p, strictly increasing in q and strictly quasi-concave, therefore $\tilde{q}(p)$ is convex, i.e. $\tilde{q}''(p) > 0$. It shall follow then that $\tilde{q}'(p_0) > \tilde{q}'(p_1)$ since $p_0 > p_1$, but that contradicts (21). Therefore the earlier supposition that $\frac{d\hat{U}}{dp}\Big|_{p_0} < 0$ is wrong. Suppose now that $\frac{d\hat{U}}{dp}\Big|_{p_0} = 0$. From (16) it then follows that

$$\hat{q}'(p_0) = -\frac{U_p(p_0,\hat{q}(p_0))}{U_q(p_0,\hat{q}(p_0))} = \frac{1}{a}.$$

Also,

$$\tilde{q}'(p_0) = -\frac{U_p(p_0,\tilde{q}(p_0))}{U_q(p_0,\tilde{q}(p_0))} = -\frac{U_p(p_0,\hat{q}(p_0))}{U_q(p_0,\hat{q}(p_0))} = \frac{1}{a}.$$

Taking $\frac{d^2\hat{U}}{dp^2}$, considering it at point p_0 and plugging in the above expression for $\hat{q}'(p_0)$ gives

$$\frac{d^2\hat{U}}{dp}\Big|_{p_0} = \frac{\lambda_H}{\lambda_M}\left(-U_{pp} - 2\frac{1}{a}U_{pq} - \frac{1}{a^2}U_{qq}\right)\Big|_{(p_0,\hat{q}(p_0))}.$$

Using (20) to get $\frac{d^2}{dp^2}\tilde{q}$, considering it at p_0 and plugging in the expression for $\tilde{q}'(p_0)$ gives

$$\frac{d^2}{dp^2}\tilde{q}\Big|_{p_0} = \frac{1}{U_q}\left(-U_{pp} - 2\frac{1}{a}U_{pq} - \frac{1}{a^2}U_{qq}\right)\Big|_{(p_0,\tilde{q}(p_0))}.$$

But $\frac{d^2}{dp^2}\tilde{q} > 0$ because iso-utility curves are convex, $U_q > 0$ because $U(p,q)$ is strictly increasing in q, $(p_0,\tilde{q}(p_0)) = (p_0,\hat{q}(p_0))$. Therefore $\frac{d^2\hat{U}}{dp^2}\Big|_{p_0} > 0$. So, $\hat{U}$ is strictly convex at p_0 with $\frac{d\hat{U}}{dp}\Big|_{p_0} = 0$. Consequently $\exists p_2 \in (p_m,p_0) : \frac{d\hat{U}}{dp}\Big|_{p_2} < 0$. As was shown before this can not be the case and therefore the supposition that $\frac{d\hat{U}}{dp}\Big|_{p_0} = 0$ is also wrong. Summarizing both arguments gives that $\frac{d\hat{U}}{dp} > 0$ for $p > p_m$. Analogues arguments give that $\frac{d\hat{U}}{dp} < 0$ for $p < p_m$. □

Lemma A.3. *For any $p \neq p_m$*

$$\frac{\partial^2\pi(p,q)}{\partial^2 q}\Big|_{(p,\hat{q}(p))} = \frac{a^2\lambda_L\lambda_M}{\lambda_H}\cdot\frac{\hat{\Pi}(U_R)}{\Pi(p,\hat{q}(p))}\left(\frac{1}{2}\frac{\hat{U}_{pq} - \frac{\hat{U}_p}{\hat{U}_q}\hat{U}_{qq}}{\hat{U}_q + a\hat{U}_p} + \frac{1}{\Pi(p,\hat{q}(p))}\right),$$

where $\hat{U}_p = \frac{\partial U(p,q)}{\partial p}\Big|_{(p,\hat{q}(p))}$ and similarly for $\hat{U}_q$, $\hat{U}_{pq}$ and $\hat{U}_{qq}$.

Proof. We prove this lemma in a straightforward way. Recollect from lemma 3 that

$$\pi(p,q) = \left(F(U(p,q)) \cdot \lambda_H + F(U(p,\hat{q}(p))) \cdot \lambda_M + \frac{\lambda_L}{2} \right) \cdot \Pi(p,q),$$

where $\Pi(p,q) = p - aq$ and $F(u)$ is given by lemma 5. If we are to differentiate $\pi(p,q)$ we need to know $\hat{q}'(p)$ and $\hat{\Pi}'(u)$. The former derivative we take from lemma 6:

$$\frac{d\hat{q}}{dp} = -\frac{\lambda_H + \lambda_M}{\lambda_M} \cdot \frac{U_p'(p,\hat{q}(p))}{U_q'(p,\hat{q}(p))} - \frac{\lambda_H}{a\,\lambda_M}. \tag{22}$$

As for the latter derivative, recollect that

$$\hat{\Pi}(u) = \Pi(\tilde{p}(u), \hat{q}(\tilde{p}(u))), \tag{23}$$

where $\tilde{p}(u)$ could be any function such that $U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u$. We'll be looking at the second order derivative of $\pi(p,q)$ at point $(p_0, \hat{q}(p_0))$ of an equilibrium curve with $p_0 \neq p_m$. For this point we can be more precise about $\tilde{p}(u)$. Indeed, from lemma A.2 we know that

$$\frac{d}{dp} U(p, \hat{q}(p)) \neq 0 \quad \text{for} \quad p \neq p_m.$$

Also $U(p,\hat{q}(p))$ is twice differentiable because for $U(p,q)$ it was assumed and $\hat{q}(p)$ is itself defined by a differential equation that involves only differentiable functions. So, by an inverse function theorem there is a unique continuously differentiable $\tilde{p}(u)$ defined in the neighbourhood of $u_0 = U(p_0, \hat{q}(p_0))$ by $U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u$, with its derivative given by

$$\frac{d\tilde{p}(u)}{du} = \frac{1}{U_p(\tilde{p}(u), \hat{q}(\tilde{p}(u)) + U_q(\tilde{p}(u), \hat{q}(\tilde{p}(u)) \cdot \hat{q}'(\tilde{p}(u))} = \\ -\frac{a\lambda_M}{\lambda_H} \frac{1}{a \cdot U_p(\tilde{p}(u), \hat{q}(\tilde{p}(u)) + U_q(\tilde{p}(u), \hat{q}(\tilde{p}(u))}. \tag{24}$$

Expressions (22), (23) and (24) allow one to calculate the second order derivative of $\pi(p,q)$ in q in a straightforward way. Evaluating the resulting derivative at $(p_0, \hat{q}(p_0))$, noticing that

$$\tilde{p}(U(p_0, \hat{q}(p_0))) = p_0$$

and noticing that p_0 was chosen arbitrary just not to equal p_m immediately gives the result of the lemma. □

Lemma A.4.

$$U_{pq}(p_m, \hat{q}(p_m)) + \frac{1}{a} U_{qq}(p_m, \hat{q}(p_m)) < 0 \qquad \Rightarrow \qquad [p_l, p_h] = [p_l, p_m],$$

$$U_{pq}(p_m, \hat{q}(p_m)) + \frac{1}{a} U_{qq}(p_m, \hat{q}(p_m)) > 0 \qquad \Rightarrow \qquad [p_l, p_h] = [p_m, p_h].$$

In other words, either the equilibrium lies in the segment to the left of p_m where higher prices signal lower utility or the equilibrium lies in the segment to the right of p_m where higher prices signal higher utility.

Proof. Consider the case when $U_{pq}(p_m, \hat{q}(p_m)) + \frac{1}{a}U_{qq}(p_m, \hat{q}(p_m)) < 0$. Suppose $p_h > p_m$. Then we can consider the limit of $\frac{\partial^2 \pi(p,q)}{\partial^2 q}\Big|_{(p,\hat{q}(p))}$ as p approaches p_m from the right. To do so let us start with the limit of $-\frac{\hat{U}_p}{\hat{U}_q}$. By the definition of p_m and by lemma A.1

$$-\frac{U_p(p_m, \hat{q}(p_m))}{U_q(p_m, \hat{q}(p_m))} = \frac{1}{a}.$$

By lemma A.2

$$\frac{d}{dp}U(p, \hat{q}(p)) > 0 \quad \text{for} \quad p > p_m.$$

Taking the derivative shows that this condition is equivalent to

$$-\frac{U_p(p, \hat{q}(p))}{U_q(p, \hat{q}(p))} > \frac{1}{a} \quad \text{for} \quad p > p_m.$$

Moreover, $-\frac{\hat{U}_p}{\hat{U}_q}$ is continuous in p. Therefore we have that

$$-\frac{\hat{U}_p}{\hat{U}_q} \downarrow \frac{1}{a} \quad \text{as} \quad p \downarrow p_m.$$

Hence

$$\lim_{p\downarrow p_m} \frac{\partial^2 \pi(p,q)}{\partial^2 q}\Big|_{(p,\hat{q}(p))} = \lim_{p\downarrow p_m} \frac{a^2\lambda_L\lambda_M}{\lambda_H} \cdot \frac{\hat{\Pi}(U_R)}{\Pi(p,\hat{q}(p))}\left(\frac{1}{2a\hat{U}_q}\frac{\hat{U}_{pq} - \frac{\hat{U}_p}{\hat{U}_q}\hat{U}_{qq}}{\frac{1}{a} - \left(-\frac{\hat{U}_p}{\hat{U}_q}\right)} + \frac{1}{\Pi(p,\hat{q}(p))}\right) = +\infty. \tag{25}$$

The sign comes from the preceding discussion and from the observation that $\hat{\Pi}(U_R)$, $\Pi(p, \hat{q}(p))$ and U_q are all strictly positive. But (25) contradicts the necessary condition that $\frac{\partial^2 \pi(p,q)}{\partial^2 q}\Big|_{(p,\hat{q}(p))} \leq 0$ for all $p \in [p_l, p_h]$. Therefore if there is an exact signalling equilibrium it should be that $p_h \leq p_m$.[7] But $p_m \in [p_l, p_h]$ (lemma A.1), hence $p_m = p_h$. Analogous arguments hold for $U_{pq}(p_m, \hat{q}(p_m)) + \frac{1}{a}U_{qq}(p_m, \hat{q}(p_m)) > 0$. □

Lemma A.5.

$$U_{pq}(p_m, \hat{q}(p_m)) + \frac{1}{a}U_{qq}(p_m, \hat{q}(p_m)) \gtrless 0 \quad \Leftrightarrow \quad g'(p_m) \gtrless \frac{1}{a}.$$

Proof. Writing down the necessary conditions for the optimization problem that defines $g(p)$ gives

$$-\frac{U_p(p, g(p))}{U_q(p, g(p))} = \frac{1}{a}.$$

[7] In this case we can not consider a limit from the right and the contradiction does not hold.

Or, equivalently,

$$aU_p(p, g(p)) + U_q(p, g(p)) = 0. \tag{26}$$

Differentiating (26) in p and rearranging the terms gives

$$g'(p) = -\frac{U_{pp} + \frac{1}{a}U_{qp}}{U_{pq} + \frac{1}{a}U_{qq}} = \frac{-U_{pp} - 2\frac{1}{a}U_{pq} - \frac{1}{a^2}U_{qq}}{U_{pq} + \frac{1}{a}U_{qq}} + \frac{1}{a}, \tag{27}$$

where $U_{pp} = \frac{\partial^2 U(p,q)}{\partial p^2}\Big|_{(p,g(p))}$, and so on. Consider now an iso-utility curve going through (p_m, q_m). Namely, consider $\tilde{q}(p)$ defined by

$$U(p, \tilde{q}(p)) = U(p_m, q_m).$$

Twice differentiating this expression, evaluating it at (p_m, q_m), noticing that

$$\tilde{q}'(p_m) = -\frac{U_p(p_m, \tilde{q}(p_m))}{U_q(p_m, \tilde{q}(p_m))} = \frac{1}{a}$$

due to the definition of (p_m, q_m), and rearranging the terms gives

$$\tilde{q}''(p_m) = \frac{1}{U_q}\left(-U_{pp} - 2\frac{1}{a}U_{pq} - \frac{1}{a^2}U_{qq}\right)\Big|_{(p_m, \tilde{q}(p_m))}. \tag{28}$$

Iso-utility curves are strictly convex (assumption 1), so $\tilde{q}''(p_m) > 0$. Also, (p_m, q_m) belongs to the contract curve $g(p)$, to the equilibrium curve $\hat{q}(p)$ and to the iso-utility curve $\tilde{q}(p)$, so $q_m = g(p_m) = \hat{q}(p_m) = \tilde{q}(p_m)$. So, we can use $U_{pp}(p_m, g(p_m))$, $U_{pp}(p_m, \hat{q}(p_m))$ and $U_{pp}(p_m, \tilde{q}(p_m))$ and the others interchangeably. But then the statement of the lemma readily follows from (27), (28), from $\tilde{q}''(p_m) > 0$ and from $U_q(p_m) > 0$. □

Theorem 3. *Consider an arbitrary strictly increasing, strictly convex and twice differentiable equilibrium curve $\hat{q}(p)$ defined over $[p_l, p_h]$ and satisfying* (11) *or* (12) *or both. Then there exist such a utility function $U(p, q)$ satisfying assumption 1, such parameters $(U_R, \lambda_H, \lambda_M, \lambda_L, a)$ and such out-of-equilibrium beliefs that there will be a corresponding exact signalling equilibrium, i.e one that has $\hat{q}(p)$ as its equilibrium curve.*

Proof. We only discuss how to find such parameters of our model as to get an equilibrium where *lower* prices signal higher utility. Construction of an equilibrium where *higher* prices signal higher utility is analogous. To prove the theorem we have to find $(U_R, \lambda_H, \lambda_M, \lambda_L, a)$, $U(p, q)$ and out-of-equilibrium beliefs such that a) $U(p, q)$ satisfies assumption 1, b) the resulting equilibrium curve is precisely $\hat{q}(p)$ and the resulting boundary points are precisely p_l and p_h, c) the expected profits $\pi(p, q)$ attain their maximum over the equilibrium curve $\hat{q}(p)$. We proceed as follows. First, we choose some specific parameters $(U_R, \lambda_H, \lambda_M, \lambda_L, a)$ and we choose a specific utility function $U(p, q)$. Second, we show that a) and b) hold for those parameters and utility function. Third, we choose some specific but reasonable out-of-equilibrium beliefs and we show that c) holds as well. Take

$$a = \frac{1}{\hat{q}'(p_h)}$$

and consider $\frac{d}{dp}\Pi(p,\hat{q}(p))$:

$$\frac{d}{dp}\Pi(p,\hat{q}(p)) = 1 - a\hat{q}'(p) = 1 - \frac{\hat{q}'(p)}{\hat{q}'(p_h)}.$$

As assumed, $\hat{q}(p)$ is strictly increasing and strictly convex, i.e. $\hat{q}'(p) > 0$ and $\hat{q}''(p) > 0$. Therefore $\hat{q}'(p) < \hat{q}'(p_h)$ for $p < p_h$ and, consequently, $\frac{d}{dp}\Pi(p,\hat{q}(p)) > 0$ for $p < p_h$. In other words, equilibrium per-unit profits are strictly increasing in p over $[p_l, p_h]$. Define

$$\Pi_l = \Pi(p_l, \hat{q}(p_l)), \quad \Pi_h = \Pi(p_h, \hat{q}(p_h)).$$

Take

$$\lambda_H + \lambda_M = \frac{\Pi_h - \Pi_l}{\Pi_h + \Pi_l}, \quad \lambda_L = 1 - (\lambda_H + \lambda_M) = \frac{2\Pi_l}{\Pi_h + \Pi_l}. \tag{29}$$

We choose precise values for $\lambda_H + \lambda_M$ and λ_L. As for λ_H and λ_M, they can be chosen arbitrary but with λ_M sufficiently small, more precisely, we take λ_M such that

$$\lambda_M < \left(\frac{\Pi_l}{\Pi_h}\right)^2 \frac{\Pi_h - \Pi_l}{\Pi_h + \Pi_l}. \tag{30}$$

Let

$$D = \{(p,q) : p \in [p_l, p_h], \Pi_l \le \Pi(p,q) \le \Pi_h\}.$$

To define $U(p,q)$ and to show that it satisfies assumption 1 we proceed as follows. First, we define $U(p,q)$ for $(p,q) \in D$ and we show that $U(p,q)$ satisfies assumption 1 on D. Second, we argue that $U(p,q)$ can be extended beyond D in such a way that the assumption is still satisfied. As a result we will have a utility function $U(p,q)$ that satisfies assumption 1 in general and has an analytical expression for $(p,q) \in D$. Take

$$U(p,q) = \frac{\Pi_h}{p - aq} - \frac{\lambda_M}{\lambda_H + \lambda_M}\frac{\Pi_h}{p - a\hat{q}(p)} - \frac{\lambda_H}{\lambda_H + \lambda_M} \quad \text{for} \quad (p,q) \in D \tag{31}$$

and take $U_R = 0$. As the equilibrium curve $\hat{q}(p)$ was taken to be twice differentiable, $U(p,q)$ is also twice differentiable on D. Consider U_q:

$$U_q(p,q) = \frac{\Pi_h}{(p - aq)^2} \cdot a > 0.$$

Hence, $U(p,q)$ is strictly increasing in q. Next, consider U_p:

$$U_p(p,q) = -\frac{\Pi_h}{(p - aq)^2} + \frac{\lambda_M}{\lambda_M + \lambda_H}\frac{\Pi_h}{(p - a\hat{q}(p))^2}(1 - a\hat{q}'(p)). \tag{32}$$

For $(p,q) \in D$ it holds that

$$p - aq \le \Pi_h, \quad p - a\hat{q}(p) \ge \Pi_l, \quad 0 \le \hat{q}'(p) \le \frac{1}{a}.$$

Therefore

$$U_p(p,q) \le -\frac{\Pi_h}{\Pi_h^2} + \frac{\lambda_M}{\lambda_M + \lambda_H}\frac{\Pi_h}{\Pi_l^2} < 0,$$

where the last inequality follows directly from (29) and (30). Hence, $U(p,q)$ is strictly decreasing in p on D. If $U(p,q)$ is strictly decreasing in p and strictly increasing in q then it is strictly quasi-concave if and only if its iso-utility curves $\tilde{q}(p)$ are strictly convex, i.e. it should be that $\tilde{q}''(p) > 0$. To check that $\tilde{q}''(p) > 0$ we start with $\tilde{q}'(p)$:

$$\tilde{q}'(p) = -\frac{U_p(p,\tilde{q}(p))}{U_q(p,\tilde{q}(p))} = \frac{1}{a} - \frac{1}{a}\frac{\lambda_M}{\lambda_M+\lambda_H}\frac{(p-a\tilde{q}(p))^2}{(p-a\hat{q}(p))^2}(1-a\hat{q}'(p)).$$

Next,

$$\tilde{q}''(p) = \frac{2}{a}\frac{\lambda_M}{\lambda_M+\lambda_H}\frac{(p-a\tilde{q}(p))^2(1-a\hat{q}'(p))^2}{(p-a\hat{q}(p))^3}\left(1-\frac{\lambda_M}{\lambda_M+\lambda_H}\frac{p-a\tilde{q}(p)}{p-a\hat{q}(p)}\right) + \frac{\lambda_M}{\lambda_M+\lambda_H}\frac{(p-a\tilde{q}(p))^2}{(p-a\hat{q}(p))^2}\hat{q}''(p).$$

The equilibrium curve $\hat{q}(p)$ was taken to be strictly convex, so $\hat{q}''(p) > 0$. Also, on D

$$p - a\tilde{q}(p) \le \Pi_h, \quad p - a\hat{q}(p) \ge \Pi_l$$

and then

$$1-\frac{\lambda_M}{\lambda_M+\lambda_H}\frac{p-a\tilde{q}(p)}{p-a\hat{q}(p)} \ge 1-\frac{\lambda_M}{\lambda_M+\lambda_H}\frac{\Pi_h}{\Pi_l} > 1-\frac{\Pi_l}{\Pi_h} > 0,$$

where the second inequality follows directly from (29) and (30). Consequently, $\tilde{q}''(p) > 0$ and $U(p,q)$ is strictly quasi-concave.

Consider $U(p,q)$ as a map of iso-utility curves on D. These iso-utility curves, when viewed as functions of p, are strictly increasing, strictly convex and are as sufficiently smooth as to make $U(p,q)$ twice differentiable. Also, D is a convex set. Clearly then, these iso-utility curves can be extended beyond D as to still be strictly increasing, strictly convex and sufficiently smooth. Moreover, if necessary, these iso-utility curves can be made convex enough outside D so as to have each of them attain a slope of $\frac{1}{a}$ at some point. This latter condition guarantees that $\max_{(p,q)} \Pi(p,q)$ s.t. $U(p,q) \ge x$ has an inner solution.

Now we proceed with verifying b). Given $(U_R, \lambda_H, \lambda_M, \lambda_L, a)$ and given $U(p,q)$ we can solve for the equilibrium curve and for the boundary points. We denote the equilibrium curve and the boundary points that we get as a solution to the model by $\hat{q}_s(p)$ and by p_l^s, p_h^s respectively, This way we can distinguish them from the given $\hat{q}(p)$ and p_l, p_h. Then to verify b) means to verify that $\hat{q}_s(p) \equiv \hat{q}(p)$, $p_l^s = p_l$ and $p_h^s = p_h$. In general, $[p_l^s, p_h^s] = [p_l^s, p_m]$ in equilibria where lower prices signal higher utility − see theorem 1. In our case we are also looking for such an equilibrium. Hence we also choose $p_h^s = p_m$. Recollect that

$$(p_m, q_m) = \arg\max_{p,q} \Pi(p,q) \quad \text{s.t.} \quad U(p,q) \ge U_R.$$

The solution is attained when $U(p,q) = U_R$ and the necessary and sufficient conditions for this optimization problem are:

$$\begin{cases} \dfrac{\Pi_p(p,q)}{\Pi_q(p,q)} = \dfrac{U_p(p,q)}{U_q(p,q)}, \\ U(p,q) = U_R, \end{cases}$$

where sufficiency follows from the strict convexity of the problem.

Suppose optimal $(p,q) \in D$. Then using (31) and simplifying gives:

$$\begin{cases} 1 - a\hat{q}'(p) = 0, \\ \dfrac{\Pi_h}{p - aq} - \dfrac{\lambda_M}{\lambda_H + \lambda_M}\dfrac{\Pi_h}{p - a\hat{q}(p)} - \dfrac{\lambda_H}{\lambda_H + \lambda_M} = 0. \end{cases} \tag{33}$$

Given that $a = \frac{1}{\hat{q}'(p_h)}$ and that $\Pi_h = p_h - a\hat{q}(p_h)$ it is straightforward to verify that point $(p_h, \hat{q}(p_h))$ satisfies (33). So, $(p_m, q_m) = (p_h, \hat{q}(p_h))$ and $p_h^s = p_h$. From lemma 6

$$\hat{q}_s'(p) = -\frac{\lambda_H + \lambda_M}{\lambda_M}\frac{U_p(p, \hat{q}_s(p))}{U_q(p, \hat{q}_s(p))} - \frac{1}{a}\frac{\lambda_H}{\lambda_M}. \tag{34}$$

The boundary condition comes from lemma A.1: $\hat{q}_s(p)$ has to go through the point $(p_m, q_m) = (p_h, \hat{q}(p_h))$. For $p \in [p_l, p_h]$ we use (32) and (5) to rewrite (34) as

$$\hat{q}_s'(p) = \frac{1}{a} - \frac{1}{a}\left(\frac{p - a\hat{q}_s(p)}{p - a\hat{q}(p)}\right)(1 - a\hat{q}'(p)).$$

Clearly, for $p \in [p_l, p_h]$ $\hat{q}_s(p) \equiv \hat{q}(p)$ is a solution. Moreover, it is unique by the Picard's theorem. The lower bound p_l^s is implicitly defined by $U(p_l^s, \hat{q}_s(p_l^s)) = U_h$ and U_h comes from $F(U_h) = 1$. We now solve these equations. For $p \in [p_l, p_h]$

$$U(p, \hat{q}_s(p)) = U(p, \hat{q}(p)) = \frac{\lambda_H}{\lambda_H + \lambda_M}\left(\frac{\Pi_h}{p - a\hat{q}(p)} - 1\right). \tag{35}$$

From lemma 5

$$F(u) = \frac{1}{2}\frac{\lambda_L}{\lambda_H + \lambda_M}\left(\frac{\Pi_h}{\hat{\Pi}(u)} - 1\right), \tag{36}$$

where

$$\hat{\Pi}(u) = \tilde{p}(u) - a\hat{q}(\tilde{p}(u))$$

and $\tilde{p}(u)$ is implicitly defined by

$$U(\tilde{p}(u), \hat{q}(\tilde{p}(u))) = u.$$

Substituting p with $\tilde{p}(u)$ in (35) and the resulting $\tilde{p}(u) - a\hat{q}(\tilde{p}(u))$ with $\hat{\Pi}(u)$ and then comparing the outcome with (36) gives

$$F(u) = \frac{1}{2}\frac{\lambda_L}{\lambda_H} \cdot u.$$

Hence $U_h = 2\frac{\lambda_H}{\lambda_L}$. Suppose $p_l^s \in [p_l, p_h]$, then using (29) and (30) we can rewrite $U(p_l^s, \hat{q}(p_l^s)) = U_h$ as

$$p_l^s - a\hat{q}(p_l^s) = \Pi_l = p_l - a\hat{q}(p_l).$$

Clearly then, $p_l^s = p_l$ is a solution. Moreover, it is unique because $U(p, \hat{q}(p))$ is strictly monotone in p (lemma A.2).

Next, we have to verify c), i.e. we have to verify that the expected profits $\pi(p,q)$ attain their maximum over $\hat{q}(p)$. When the opponent is playing the equilibrium strategy

$$\pi(p,q) = \left(F(U(p,q)) \cdot \lambda_H + F(\hat{U}(p)) \cdot \lambda_M + \frac{\lambda_L}{2}\right)\Pi(p,q), \tag{37}$$

where $\hat{U}(p)$ stands for the utility that partially informed consumers expect to receive given that the price is p. For $p \in [p_l, p_h]$ we have that

$$\hat{U}(p) = U(p, \hat{q}_s(p)) = U(p, \hat{q}(p)).$$

Since $U(p, \hat{q}(p))$ is strictly decreasing in p for $p \in [p_l, p_h]$ we can choose and we choose such out-of-equilibrium beliefs that $\hat{U}(p)$ is decreasing in p for $p \in \mathbb{R}$.[8] Define

$$\begin{aligned}
S_C &= \{(p,q) : p \in [p_l, p_h], U_R \le U(p,q) \le U_h\},\\
S_B &= \{(p,q) : U(p,q) < U_R\},\\
S_L &= \{(p,q) : p < p_l, U_R \le U(p,q) \le U_h\},\\
S_R &= \{(p,q) : p > p_h, U_R \le U(p,q) \le U_h\},\\
S_T &= \{(p,q) : U(p,q) > U_h\}.
\end{aligned}$$

Clearly, $\bigcup_x S_x = \mathbb{R}^2$. We consider $\pi(p,q)$ over each of these regions in turn.

Region $\mathbf{S_C}$. Suppose $(p,q) \in S_C$. For $p \in [p_l, p_h]$ it holds that $\Pi_l \le p - a\hat{q}(p) \le \Pi_h$. Then, using the definitions for $\lambda_H + \lambda_M$ and λ_L, it is straightforward to verify that

$$\begin{aligned}
U(p,q) \ge U_R &\Rightarrow \Pi(p,q) \le \Pi_h,\\
U(p,q) \le U_h &\Rightarrow \Pi(p,q) \ge \Pi_l.
\end{aligned}$$

Consequently, $S_C \subseteq D$. But for $(p,q) \in D$ we have an explicit expression for $U(p,q)$ and $\hat{U}(p) = U(p, \hat{q}(p))$ for $p \in [p_l, p_h]$. Also, $F(u) = \frac{1}{2}\frac{\lambda_L}{\lambda_H} \cdot u$ for $U_R \le u \le U_h$. Expanding (37) then gives:

$$\begin{aligned}
\pi(p,q) = \left(F(U(p,q)) \cdot \lambda_H + F(U(p,\hat{q}(p))) \cdot \lambda_M + \frac{\lambda_L}{2} \right) \cdot (p - aq) = \\
\frac{\lambda_L}{2} \left(U(p,q) + \frac{\lambda_M}{\lambda_H} U(p, \hat{q}(p)) + 1 \right) \cdot (p - aq) = \frac{\lambda_L}{2} \Pi_h,
\end{aligned}$$

where the last equality follows directly from the definitions of $U(p,q)$, $\lambda_H + \lambda_M$, λ_L and Π_h, Π_l, see equations (31), (29) and (5). So, the expected profits are constant for $(p,q) \in S_C$.

Region $\mathbf{S_B}$. Suppose $(p,q) \in S_B$. But then $U(p,q) < U_R$, so no consumers buy the product and

$$\pi(p,q) = 0 < \frac{\lambda_L}{2} \Pi_h.$$

Region $\mathbf{S_L}$. Suppose $(p,q) \in S_L$. This implies $U_R \le U(p,q) \le U_h$. Let

$$\begin{aligned}
q_u &= \hat{q}(p_l),\\
q_b &= \frac{1}{a} \left(p_l - \frac{(\lambda_H + \lambda_M)\Pi_l \Pi_h}{\lambda_M \Pi_h + \lambda_H \Pi_l} \right).
\end{aligned}$$

[8] These are reasonable out-of-equilibrium beliefs as they depend upon p in the same direction as equilibrium beliefs do.

Then it directly follows from the definitions of $U(p,q)$, U_R, $\lambda_H+\lambda_M$, λ_L and Π_h, Π_l that $U(p_l,q_u)=U_h$ and $U(p_l,q_b)=U_R$. Moreover, $U(p_l,q)$ is continuous in q and therefore there exists q^* such that

$$U(p_l,q^*)=U(p,q). \tag{38}$$

Consequently,

$$F(U(p,q))=F(U(p_l,q^*)). \tag{39}$$

Since $p<p_l$ and since $\hat{U}(p)$ is decreasing in p, $\hat{U}(p)\geq\hat{U}(p_l)$. But $\hat{U}(p_l)=U(p_l,\hat{q}(p_l))=U_h$ and $F(u)=1$ for $u\geq U_h$. So,

$$F(\hat{U}(p))=F(\hat{U}(p_l)). \tag{40}$$

Given (38), we can take an iso-utility curve $\tilde{q}(p)$ going through points (p,q) and (p_l,q^*). From (5) we have that

$$\tilde{q}'(p_l)=\frac{1}{a}-\frac{1}{a}\frac{\lambda_M}{\lambda_M+\lambda_H}\frac{(p_l-a\tilde{q}(p_l))^2}{\Pi_l^2}(1-a\hat{q}'(p_l)).$$

As $\hat{q}'(p_l)<\frac{1}{a}$ it follows that $\tilde{q}'(p_l)<\frac{1}{a}$. Utility $U(p,q)$ satisfies assumption 1 and therefore $\tilde{q}''(p)>0$. So, $\tilde{q}'(p)<\frac{1}{a}$ for all $p\leq p_l$. Consequently,

$$\frac{d}{dp}\Pi(p,\tilde{q}(p))=1-a\tilde{q}'(p)>0\quad\text{for all}\quad p\leq p_l.$$

As $p<p_l$ we then have that

$$\Pi(p,q)<\Pi(p_l,q^*). \tag{41}$$

Bringing together (39), (40) and (41) and noticing that $(p_l,q^*)\in S_C$ gives:

$$\pi(p,q)=\left(F(U(p,q))\cdot\lambda_H+F(\hat{U}(p))\cdot\lambda_M+\frac{\lambda_L}{2}\right)\Pi(p,q)<$$
$$\left(F(U(p_l,q^*))\cdot\lambda_H+F(\hat{U}(p_l))\cdot\lambda_M+\frac{\lambda_L}{2}\right)\Pi(p_l,q^*)=\frac{\lambda_L}{2}\Pi_h.$$

Region $\mathbf{S_R}$. Suppose $(p,q)\in S_R$. This case is analogous to the previous one and we also get that

$$\pi(p,q)<\frac{\lambda_L}{2}\Pi_h.$$

Region $\mathbf{S_T}$. Suppose $(p,q)\in S_T$, i.e. $U(p,q)>U_h$. Given that $U(p,q)$ satisfies assumption 1 there exists $q^*<q$ such that $U(p,q^*)=U_h$. As $F(u)=1$ for $u\geq U_h$ we have that $F(U(p,q))=F(U(p,q^*))$. Trivially, $\Pi(p,q)<\Pi(p,q^*)$. Therefore,

$$\pi(p,q)=\left(F(U(p,q))\cdot\lambda_H+F(\hat{U}(p))\cdot\lambda_M+\frac{\lambda_L}{2}\right)\Pi(p,q)<$$
$$\left(F(U(p,q^*))\cdot\lambda_H+F(\hat{U}(p))\cdot\lambda_M+\frac{\lambda_L}{2}\right)\Pi(p,q^*)=\pi(p,q^*).$$

But $(p,q^*)\in S_L\cup S_C\cup S_R$, so $\pi(p,q^*)\leq\frac{\lambda_L}{2}\Pi_h$ and $\pi(p,q)<\frac{\lambda_L}{2}\Pi_h$. Given that $\pi(p,q)=\frac{\lambda_L}{2}\Pi_h$ for $(p,q)\in S_C$, that $\pi(p,q)<\frac{\lambda_L}{2}\Pi_h$ for $(p,q)\notin S_C$ and that the equilibrium curve $\hat{q}(p)$ belongs to S_C, we have that the profits attain their maximum over $\hat{q}(p)$. □