Option Pricing with Transaction Costs: the Superhedging Approach[1]

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In this paper I address the problem of pricing options in the presence of transaction costs. A known approach to this problem is a so-called superhedging approach. I suggest an idea of hedging sets and, based on it, a new algorithm to implement the superhedging approach. As far as I know, only European options were discussed in the literature on pricing derivatives when there are transaction costs. The proposed algorithm, however, can be applied to price American options as well. It is also more efficient compared with the previous methods.

The superhedging approach is based on the problem of finding the cheapest portfolio now for which there is a trading strategy with consumption such that for every possible outcome the value of the portfolio at the time of the option exercise is not less than the value of the option (when the short position in the option is considered).

A notion of a hedging set is introduced in the article to help to solve this problem. A discrete time economy is considered with price movements of an underlying asset being modeled by a recombining tree. At the current tree node a hedging set is defined as a set of portfolios that can hedge the considered option in the sense of superhedging. It is easy to build hedging sets for every tree node at the time of the option exercise. I also show how to build hedging sets for the current tree nodes given hedging sets of the next tree nodes when there are proportional transaction costs. Thus the solution to the stated problem comes from building hedging sets backward the tree and then finding the cheapest portfolio in the first hedging set.

I discuss in detail the idea of hedging sets and how to price European and American options under the presence of proportional transaction costs. Appropriate numerical examples are also given.

I believe that the idea of hedging sets is versatile as it allows using a tree of any type to model price movements of an underlying asset when pricing and hedging derivatives using the superhedging approach. Also, multidimensional hedging sets may allow pricing and hedging derivatives that depend on several random factors.

Introduction

One of the open tasks suggested by financial mathematics is the question of pricing and hedging derivatives under the presence of transaction costs. Putting the

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[1] This is a shorter version of the original article “Option Pricing and Hedging with Transaction Costs: the Superhedging Approach” // Economics Journal of Higher School of Economics, 2004, vol. 8, no. 4., p. 491-519.

This work in full detail gives the new solution to the superhedging problem as it is proposed in the original article. The original article also includes in addition to some miscellaneous details a comprehensive description of the superhedging problem, discusses asymmetric proportional transaction costs and addresses the question of hedging.

[2] The author thanks Alexei Shvedov for helpful comments on the original Russian version of the text.

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problem broader it is pricing and hedging derivatives in incomplete markets as main reasons of incompleteness can be modeled by transaction costs. For example, transaction costs may count not only for some trade commissions, but also for bid-ask spreads, for assets liquidity, for days when trade in some assets is not possible. That is less important for the developed markets where the most of transaction costs are low for large institutions, but for the developing markets the significance of the question has the following reasoning. Transaction costs are high in the developing markets, and if financial institutions cannot accurately manage their instruments that depend on the transaction costs, the issue of these instruments is impossible. Though the presence and the variety of derivative instruments have significant positive effect on the economy development.

Taking transaction costs into derivative's pricing models may involve different techniques. This article helps those who expand Cox-Ross-Rubinstein approach [5] to deal with transaction costs. Briefly the problem of pricing options with transaction costs using trees was addressed by Robert Merton in his book [9, Chapter 14, Section 14.2]. Later Boyle and Vorst expanded Merton's approach to a multiperiod model and introduced pricing both the long and the short positions in options.

When there are no transaction costs an arbitrage-free price of an option is unique. If there are transaction costs an interval of arbitrage-free prices arises as the hedging that is necessary to realize the arbitrage opportunities currently involves these transaction costs. Ceteris paribus the task is to find a tighter estimation of this interval. Here the superhedging approach presented in articles [2] and [6] works better than Boyle and Vorst approach: it gives a tighter interval estimation of arbitrage-free prices. Article [2] discusses some special cases; for European options and proportional transaction costs Edirisinghe, Naik and Uppal present in their later article [6] an algorithm of the superhedging approach implementation that involves grid-search and solving equations numerically.

This article presents a new algorithm to solve the superhedging problem, which is more visual than shown in [6]; it provides an exact solution and what is more important it is more general. The algorithm allows pricing both European and American options, it can be expanded to deal not only with proportional transaction costs, but also with constant and progressive costs and it can be implemented for trees of any type.

This article has the following structure. Firstly, the superhedging problem is stated and the new algorithm is proposed as a solution of this problem. Secondly, an example of pricing European options using the algorithm proposed is given. As a comparison the results obtained from Boyle and Vorst approach are also provided. Thirdly, the algorithm is further expanded to deal with American options. Appropriate numerical examples are given.

A Geometrical Approach to the Superhedging Problem

Next the superhedging problem is stated (see [2], [6]). A binominal non-recombining tree is considered, the notations are: T denotes the number of periods, t denotes a particular period, $t \in \{1,\ldots,T\}$, i - a particular state for the given t,

$i \in \{1,\dots,2^{t-1}\}$ since the tree does not recombine, $j(i)$ - a particular state of the next period given some state i for the current period, $j(i) \in \{2i-1, 2i\}$. $P(t,i)$ denotes prices of an underlying at the correspondent nodes of the tree. $C(T,i)$ are prices of an option at the time of its maturity. (a,b) is a portfolio, a stands for the quantity of the underlying, b stands for the amount of money, either own or borrowed. $a(t,i)$ and $b(t,i)$ are the parameters of the portfolio at the correspondent nodes of the tree. k stands for proportional transaction costs, it is assumed that whenever the underlying is traded the volume of trade is subject to commission of rate k.

The top limit of the interval of the arbitrage-free prices of the option, denoted by $\overline{C}$, is given by

(1) $$P(T,i)\cdot a(T,i) + b(T,i) \geq C(T,i) \quad \text{for} \quad i = \overline{1, 2^{T-1}},$$

(2) $$P(t+1, j(i))\cdot a(t,i) + (1+r)\cdot b(t,i) \geq$$
$$\geq P(t+1, j(i))\cdot a(t+1, j(i)) + b(t+1, j(i)) + k\cdot P(t+1, j(i))\cdot |a(t+1, j(i)) - a(t,i)|$$
$$\text{for} \quad t = \overline{1, T-1}, \quad i = \overline{1, 2^{t-1}} \quad \text{and} \quad j(i) \in \{2i-1, 2i\},$$

(3) $$\overline{C} = \min_{\{a,b\}}[P(1,1)\cdot a(1,1) + b(1,1)],$$

where $\min_{\{a,b\}}$ denotes the minimum over every $a(t,i)$ and $b(t,i)$ subject to (1), (2).

If the market option price is equal to or higher than $\overline{C}$, equations (1)-(3) assume that there is a strategy with consumption that starts from taking the short position in the option and results into the ability to pay the strike price. As there is a chance that some money is left (a strategy with consumption is considered) it creates an arbitrage opportunity. So, there to be no arbitrage opportunities the option should be priced strictly less than $\overline{C}$.

The bottom limit of the interval of the arbitrage-free prices of the option, denoted by $\underline{C}$, is given by

(4) $$P(T,i)\cdot a(T,i) + b(T,i) \geq -C(T,i) \quad \text{for} \quad i = \overline{1, 2^{T-1}},$$

(5) $$P(t+1, j(i))\cdot a(t,i) + (1+r)\cdot b(t,i) \geq$$
$$\geq P(t+1, j(i))\cdot a(t+1, j(i)) + b(t+1, j(i)) + k\cdot P(t+1, j(i))\cdot |a(t+1, j(i)) - a(t,i)|$$
$$\text{for} \quad t = \overline{1, T-1}, \quad i = \overline{1, 2^{t-1}} \quad \text{and} \quad j(i) \in \{2i-1, 2i\},$$

(6) $$\underline{C} = -\min_{\{a,b\}}[P(1,1)\cdot a(1,1) + b(1,1)].$$

If the market price of the option is equal to or less than $\underline{C}$, equations (4)-(6) assume that there is a strategy with consumption that starts from taking the long position in the option and results into the ability to pay back the loan. Since there is a chance that some money is left it creates an arbitrage opportunity. So, there to be no arbitrage opportunities the option should be priced strictly more than $\underline{C}$.

The solution to (1)-(6) gives the required interval $(\overline{C};\underline{C})$. There are no arbitrage opportunities if the market price of the option belongs to it.

The portfolio parameters $a(t,i)$ and $b(t,i)$, that give the solution, are path-dependent, so the non-recombining tree is considered (see [6]).

This article suggests a new approach to solving the superhedging problem stated by equations (1)-(3) and (4)-(6). The relevant algorithm is presented below. Equations (1)-(3) are particularly described and only a final solution is given to equations (4)-(6) as they are similar.

The algorithm is presented being divided into four stages, and the third most interest stage is additionally divided into several steps.

Stage 1. Construction of a binominal tree

This time a recombining binominal tree is built. The notations are as previous, though $i \in \{1,\dots,t\}$ and $j(i) \in \{i, i+1\}$ since the tree is now a recombining one. It is noted that the tree recombines regarding the prices of an underlying asset, i.e. regarding $P(t,i)$.

If some portfolio parameters ($a(t,i)$ and $b(t,i)$ for every t and i) solve the superhedging problem then the tree cannot be recombined regarding them as it has been stated earlier. The presented approach does not face this problem as it works in a different way – rather than to find the optimum $a(t,i)$ and $b(t,i)$ for every tree node, it operates with sets of $a(t,i)$ and $b(t,i)$ which in turn may provide the optimum $a(t,i)$ and $b(t,i)$. This central idea of the research will be further completely explained, now it should be noted that it allows considering a tree that recombines regarding $P(t,i)$ without a loss of generalization.

Stage 2. Construction of hedging sets at the option exercise moment

Here the notion of a hedging set is given. Let t and i be fixed. $a(t,i)$ denotes the quantity of an underlying asset in the portfolio for the given t and i, $b(t,i)$ denotes the amount of money in the portfolio. Then set $X(t,i)$ consisting of portfolios $(a(t,i),b(t,i))$ is defined as a hedging set if it includes every portfolio for which there is a trading strategy with consumption such that the value of the considered obligations is not greater than the value of the considered assets at moment T given any path of the economy over period $[t,T]$. When t and i are fixed set $X(t,i)$ is a subset of a two-dimensional plane, where the first axis (a) stands for the quantity of an underlying asset and the second axis (b) stands for the amount of money.

Next, hedging sets are built for every state of the economy at the moment of the option exercise. Since hedging of a short position in the option is considered (eq. (1)-(3)) a hedging set at the moment of the option exercise consists of all the portfolios which value is more than the option strike price:

(7) $$X(T,i) = \{(a,b) : P(T,i) \cdot a + b \geq C(T,i)\} \quad \text{for} \quad i = \overline{1,T}.$$

The illustration shows that hedging sets at moment T are shaped as half planes.

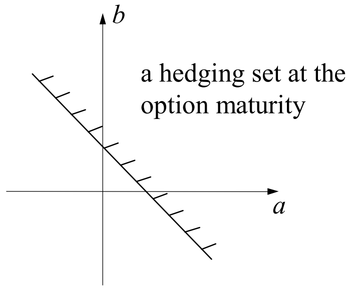

Fig. 1. A hedging set at moment T (the option exercise moment) and at state i. The borderline is given by the equation $b = C(T,i) - a \cdot P(T,i)$.

Stage 3. Construction of current moment hedging sets by the next moment hedging sets

At moment T any hedging set is an unrestricted convex polygon with the finite number of vertexes. More reasoning will reveal this property is kept also for the hedging sets that are constructed for previous moments (for $t < T$). To be more exact, at T the number of vertexes is zero but it may increase when t decreases.

Let us consider an arbitrary fragment of the binominal tree (fig. 2). Let there be defined hedging sets $A = X(t+1, i+1)$ and $B = X(t+1, i)$ at the $t+1$ time moment for i and $i+1$ states. Next, hedging set $X(t,i) = E$ is built for moment t and state i basing on hedging sets A and B (fig. 2, 3).

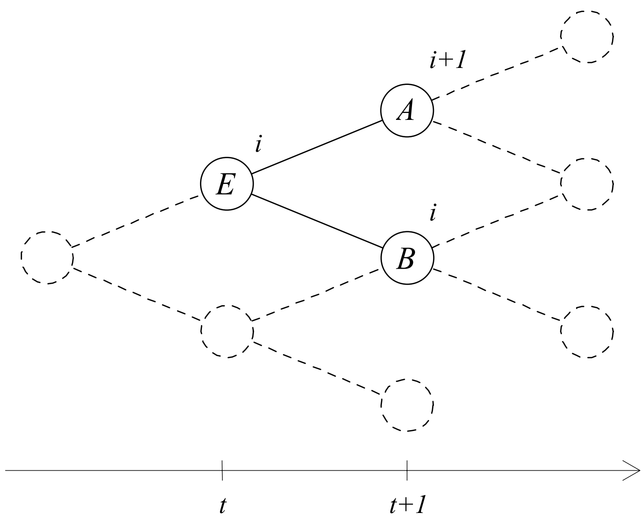

Fig. 2. A fragment of a binominal tree.

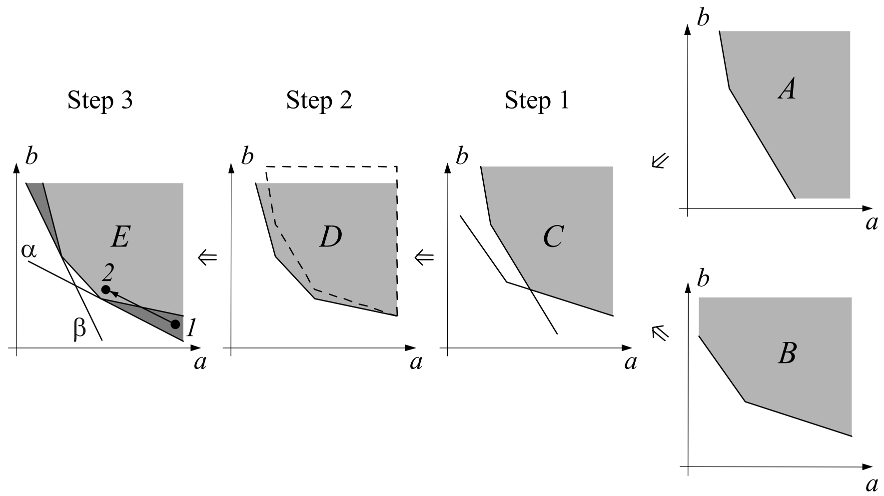

Fig. 3. Construction of current moment hedging sets by the next moment hedging sets.

Step 1. Let $C = A \cap B$. Then, if some portfolio belongs to C at moment $t+1$, it also belongs to some hedging set (A or B) at any economy state (given any i). In opposition if some portfolio does not belong to C there is always some state i at which the portfolio does not belong to the correspondent hedging set.

Step 2. Over $[t, t+1]$ the amount of money b earns interest r. Then, let us define set D so that if some portfolio belongs to C at $t+1$ then it also belongs to D at t and vise versa. This gives:

(8) $$D = \{(a,b) : (a, (1+r)b) \in C\}.$$

Step 3. Next, set D is expanded into set E by trading:

(9) $$E = \{(u,v) : P(t,i) \cdot u + v \geq P(t,i) \cdot a + b + k \cdot P(t,i) \,|\, u - a \,|, (a,b) \in D\}.$$

Then if some portfolio (u,v) belongs to E it can always be traded into some other portfolio (a,b), which belongs to D. Such trading operation is always self-financing and accounts for the proportional transaction costs given by k.

The problem of D-E expanding has a geometrical interpretation. Firstly, consider the case of selling an underlying asset, i.e. the case when $u \geq a$. For example, let portfolio (u,v) be at point 1 (fig. 3, step 3). Then an underlying can be sold and the commission (transaction costs) can be paid from the money received, thus, transforming portfolio (u,v) into a new portfolio (a,b) located at point 2. This operation is subjected to the following constraint:

(10) $$P(t,i) \cdot u + v \geq P(t,i) \cdot a + b + k \cdot P(t,i)(u - a).$$

To put it differently:

$$\frac{v-b}{u-a} \geq -(1-k)P(t,i)\,. \tag{11}$$

It means that the slope angle of vector 1-2 should be less than $-(1-k)P(t,i)$. If this condition is met, from the state of portfolio (a,b) belonging to D it follows that portfolio (u,v) belongs to set E by the definition of the latter.

The expression $-(1-k)P(t,i)$ does not depend on the coordinates of points 1 and 2, i.e. it does not depend on (u,v) and (a,b). This means that at selling an underlying expanding set D into set E is performed first by drawing a tangent to the border of set D with the slope angle equaling $-(1-k)P(t,i)$ (fig. 3, line α) and then by expanding D down to the tangent and to the right from the contact point.

At buying an underling expanding set D into set E is performed first by drawing a tangent to the border of set D with the slope angle equaling $-(1+k)P(t,i)$ (fig. 3, line β) and then by expanding D down to the tangent and to the left from the contact point.

Combining both cases (selling and buying of an underlying) gives set E expanded from set D.

The so-constructed set E is a hedging set as any portfolio from the set has a correspondent trading strategy with consumption that can bring it to the hedging set again when time goes from t to $t+1$ and the new state of the economy is either i or $i+1$ (in the first case the new hedging set is B, in the second - A). Conversely, if some portfolio is not in E then there is $j(i)$ such there is no trading strategy with consumption that can bring this portfolio to a hedging set again when time makes one step forward.

Combination of steps 1, 2, 3 gives the following expression for $X(t,i)$:

$$X(t,i) = \{(u,v) : P(t,i)\cdot u + v \geq P(t,i)\cdot a + b + k\cdot P(t,i)\,|\,u-a\,|,\\ (a,(1+r)b) \in X(t+1,i) \cap X(t+1,i+1)\}\,. \tag{12}$$

Stage 4. Determination of the top limit of an interval of arbitrage-free prices

Hedging sets can be built for moment T (eq. 7), also hedging sets for moment t can be built from the hedging sets of moment $t+1$ (eq. 12). Then the backward induction gives a hedging set for the first moment, i.e. hedging set $X(1,1)$.

If some pairs $a(t,i)$ and $b(t,i)$ satisfy constraints (1), (2) for every t and i then $(a(1,1),b(1,1)) \in X(1,1)$. And vise versa, if there is some pair $(a,b) \in X(1,1)$ then there are pairs $a(t,i)$ and $b(t,i)$ such that constraints (1), (2) are satisfied and $(a(1,1),b(1,1)) = (a,b)$.

Determination of the top limit of the interval of arbitrage-free prices, i.e., determination of $\overline{C}$ given by equation (3), is a problem of minimizing $P(1,1)\cdot a+b$ over set $X(1,1)$:

(13) $$\overline{C}=\min_{(a,b)\in X(1,1)}(P(1,1)\cdot a+b).$$

The problem can be interpreted geometrically as finding a tangent to the border of set $X(1,1)$ with the slope angle equaling $-P(1,1)$ (fig. 4).

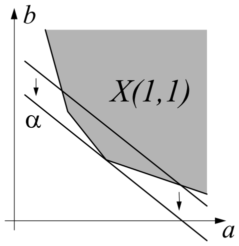

Fig. 4. Cheapest portfolio determination in set $X(1,1)$. The tangent α is given by the following equation: $b=\overline{C}-P(1,1)\cdot a$.

The complexity of the proposed algorithm requires some comments. When considering the class of set-level operations (union, intersection, and expansion in terms of the article) the algorithm has a quadratic complexity. It means that the number of the steps of this class required to price an option is a quadratic function of the number of moments (the number of the required steps can be expressed as $c_1+c_2T+c_3T^2$ where c_i does not depend on T). Also, most of the set-level operations have linear algorithms: the number of the arithmetic operations required to perform the given algorithm can be expressed as a linear function of the number of the vertexes of the sets involved (it should be noted that hedging sets have polygon forms as far as proportional transaction costs are considered). Combination of two-dimensional sets, intersection, tangent finding are the examples of operations that have linear algorithms. Then, if the number of the vertexes of the sets used during calculations does not depend on T, the algorithm in whole has a quadratic complexity, but generally the number of vertexes increases regarding T, so the complexity of the algorithm is higher then just quadratic. Still calculations show that it works much faster then exponential algorithms.

Next, the proposed option pricing algorithm under the presence of proportional transaction costs is stated formally once again. The formulas for finding the top limit of the interval of arbitrage-free prices are given below:

(14) $$X(T,i)=\{(a,b):P(T,i)\cdot a+b\geq C(T,i)\}\quad \text{for}\quad i=\overline{1,T},$$

(15) $$X(t,i)=\{(u,v):P(t,i)\cdot u+v\geq P(t,i)\cdot a+b+k\cdot P(t,i)\,|u-a|,$$
$$(a,(1+r)b)\in X(t+1,i)\cap X(t+1,i+1)\}\quad \text{for}\quad t=\overline{1,T-1}\quad \text{and}\quad i=\overline{1,t},$$

(16) $$\overline{C} = \min_{(a,b)\in X(1,1)} (P(1,1)\cdot a + b)\,.$$

The formulas for the bottom limit are:

(17) $$X(T,i) = \{(a,b) : P(T,i)\cdot a + b \ge -C(T,i)\} \quad \text{for} \quad i = \overline{1,T}\,,$$

(18) $$X(t,i) = \{(u,v) : P(t,i)\cdot u + v \ge P(t,i)\cdot a + b + k\cdot P(t,i)\,|\,u-a\,|,$$
$$(a,(1+r)b) \in X(t+1,i)\cap X(t+1,i+1)\} \quad \text{for} \quad t = \overline{1,T-1} \quad \text{and} \quad i = \overline{1,t}\,,$$

(19) $$\underline{C} = -\min_{(a,b)\in X(1,1)} (P(1,1)\cdot a + b)\,.$$

Let

(20) $$\alpha(t,i) = \frac{(1+r)P(t,i) - P(t+1,i)}{P(t+1,i+1) - P(t+1,i)}\,.$$

Then, if $0 \le \alpha(t,i) \le 1$ for every $t = \overline{1,T-1}$ and for every $i = \overline{1,t}$ the top limit of the interval is not less than the bottom limit: $\overline{C} \ge \underline{C}$ (see appendix 1, lemma 2).

An Example of Pricing European Options

The example of pricing a European call is given below. The geometrical Brownian motion is taken to model price behavior of an underlying asset and Cox-Ross-Rubinstein approach is used to determine the prices of an underlying at the recombining tree nodes [5]:

(21) $$P(t,i) = P(1,1)\cdot \exp\left(\sigma\sqrt{\frac{\mathbf{T}}{T}}(2i-t-1)\right) \quad \text{for} \quad t = \overline{2,T} \quad \text{and} \quad i = \overline{1,t}\,,$$

where $\mathbf{T}$ is the time to maturity, in years, and σ is the volatility of the underlying. One period interest rate r is set by

(22) $$r = \exp(R\cdot \mathbf{T}/T)\,,$$

where R is the annual interest rate with continuous compounding.

The price of the option at the exercise moment is

(23) $$C(T,i) = \max(P(T,i) - K, 0)\,,$$

where K is the strike price.

Placing equations (21)-(23) into equations (14)-(16) gives the top limit of the interval of arbitrage-free prices of the option. Placing (21)-(23) into (17)-(19) gives the bottom limit.

Articles [4] and [10] give examples of pricing options using Boyle and Vorst approach, article [6] gives examples of pricing options using the superhedging approach when $T \le 19$. The accuracy of the computation algorithms used in the current work (the algorithm for Boyle and Vorst approach and the algorithm for the superhedging approach) was checked by comparing their results with the provided outcomes in the above mentioned articles. In the current work the computation algorithm used to implement the superhedging approach is the algorithm presented in the previous section.

Table 1 gives an example of intervals of arbitrage-free prices of European call options found using both Boyle and Vorst approach and the superhedging approach.

Table 1. Pricing European Options.

Let us consider a call option. The first interval is $(\underline{C}, \overline{C})$ received on the basis of Boyle and Vorst approach, the second is the interval received due to the superhedging approach. The percentage key figure is the interval lengths ratio. T denotes the number of periods; the current price of the underlying is $P(1,1) = 100$; $K = 100$ – the strike price; $R = 0.1$ – annual interest rate; σ denotes the volatility of the underlying; k denotes the rate of the commission taken at traded underlying; **T** = 1 – time to maturity (one year).

$\sigma = 0.2$

T	$k = 0.00$	$k = 0.01$	$k = 0.02$	$k = 0.03$
16	8.042 - 8.042 8.042 - 8.042 n.a.	6.147 – 9.552 6.322 - 9.460 92.18%	3.212 - 10.861 3.907 - 10.700 88.81%	n.a. - 12.035 2.334 - 11.820 n.a.
32	8.204 - 8.204 8.204 - 8.204 n.a.	5.500 – 10.208 5.605 - 10.153 96.62%	n.a. - 11.880 2.593 - 11.786 n.a.	n.a. - 13.347 1.917 - 13.223 n.a.
64	8.158 - 8.158 8.158 - 8.158 n.a.	3.661 – 10.883 3.844 - 10.845 96.93%	n.a. - 13.043 1.085 - 12.980 n.a.	n.a. - 14.889 0.468 - 14.793 n.a.
128	8.186 - 8.186 8.186 - 8.186 n.a.	n.a. – 11.831 1.795 - 11.807 n.a.	n.a. - 14.570 0.889 - 14.527 n.a.	n.a. - 16.858 0.889 - 16.787 n.a.

$\sigma = 0.3$

T	$k = 0.00$	$k = 0.01$	$k = 0.02$	$k = 0.03$
16	12.248 - 12.248 12.248 - 12.248 n.a.	10.529 - 13.755 10.649 - 13.665 93.49%	8.468 - 15.116 8.768 - 14.952 93.03%	5.696 - 16.367 6.370 - 16.142 91.58%
32	12.085 - 12.085 12.085 - 12.085 n.a.	9.549 – 14.167 9.641 - 14.106 96.66%	5.868 - 15.976 6.281 - 15.868 94.85%	n.a. - 17.597 3.010 - 17.452 n.a.
64	12.148 - 12.148 12.148 - 12.148 n.a.	8.385 – 14.969 8.457 - 14.930 98.33%	n.a. - 17.323 2.875 - 17.256 n.a.	n.a. - 19.382 1.969 - 19.293 n.a.
128	12.133 - 12.133 12.133 - 12.133 n.a.	6.138 – 15.963 6.210 - 15.936 99.00%	n.a. - 19.004 1.510 - 18.961 n.a.	n.a. - 21.597 0.950 - 21.534 n.a.

$\sigma = 0.4$

T	$k = 0.00$	$k = 0.01$	$k = 0.02$	$k = 0.03$
16	16.262 - 16.262 16.262 - 16.262 n.a.	14.595 - 17.771 14.706 - 17.681 93.68%	12.709 - 19.162 12.966 - 18.996 93.46%	10.483 - 20.458 10.954 - 20.227 92.96%
32	16.126 - 16.126 16.126 - 16.126 n.a.	13.712 - 18.211 13.796 - 18.150 96.77%	10.747 - 20.073 10.964 - 19.962 96.49%	6.491 - 21.769 7.037 - 21.617 95.44%
64	16.003 - 16.003 16.003 - 16.003 n.a.	12.412 - 18.880 12.499 - 18.839 98.02%	6.986 - 21.352 7.406 - 21.279 96.57%	n.a. - 23.547 2.675 - 23.450 n.a.
128	16.071 - 16.071 16.071 - 16.071 n.a.	10.714 - 19.981 10.766 - 19.954 99.14%	n.a. - 23.198 2.746 - 23.152 n.a.	n.a. - 25.982 1.949 - 25.922 n.a.

The table illustrates the intervals of arbitrage-free prices derived using either Boyle and Vorst approach or the superhedging approach significantly expands regarding the number of periods (regarding T). The reasons are obvious. The use of the geometrical Brownian motion to model prices of an underlying allows the price to be of any altitude at the moment of an option exercise (if $T \to \infty$), although the very high and very low values are less probable. When transaction costs are not present and T increases the hedging is achieved by increasing the frequency of portfolio adjustments, but when transaction costs are present this comes to losses, which in turn leads to escalating of intervals of arbitrage-free prices.

The table also illustrates the superhedging approach always gives tighter intervals than Boyle and Vorst approach, generally the win is about 5%. On the one hand the difference diminishes with the increase of T and Boyle and Vorst approach has a simpler computational algorithm than the superhedging approach does. On the other hand Boyle and Vorst approach has a limited applicability – it is applicable only to binominal trees and requires some technical restrictions on input variables (n.a. in the table illustrates the cases when their approach does not work) – and the superhedging approach always gives a solution and it can be used with trees of any type.

Pricing American Options

At the prior sections the algorithm for the superhedging approach for the case of European options was presented. In this section the algorithm is expanded to also embrace American options.

Determining the top limit of an interval of arbitrage-free prices for a European option involves hedging the short position in the option. The difference between American and European options is that the former can be exercised earlier. So, when hedging the short position in an American option counterparty may exercise it any time. Let us back to figures 2 and 3. Given hedging sets for the $t+1$ moment and for the i and $i+1$ states (hedging sets B and A respectively) a hedging set for the t moment and for the i state is built – that is hedging set E. Then if a portfolio belongs to E

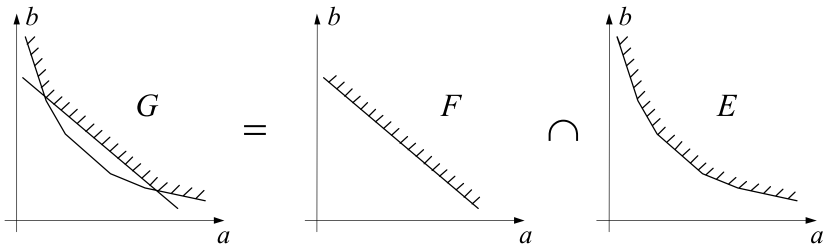

Fig. 5. Construction of a hedging set that counts for a possibility of an early exercise of an option.

there is a correspondent trading strategy with consumption that can bring it into a new hedging set when one time period passes. Next, let us consider figure 5.

If the counterparty exercises the option at moment t and state i (the correspondent payment is denoted by $C(t,i)$) the portfolio should belong to set F defined as following:

(24) $$F = \{(a,b) : P(t,i) \cdot a + b \geq C(t,i)\}.$$

This guarantees the value of the portfolio can meet the obligation arising from the short position.

Also the portfolio should belong to set E. Together this means that the hedging set for the current moment and state (for t and i), let us denote it as G, is constructed by taking an intersection of sets F and E. The rest of the algorithm remains the same: firstly, hedging sets for the last moment are constructed, secondly, hedging sets for the other moments are constructed going backward the tree and, thirdly, the cheapest portfolio is determined in the first hedging set and its price taken as the top limit of the interval of the arbitrage-free prices of the option.

Determining the bottom limit of an interval of arbitrage-free prices for a European option involves hedging the long position in the option. American option pricing is similar, but an American option can be exercised at any moment. But in the case of the long position the option exercise right belongs to the option holder. Let E and F be the same hedging sets as before except they are the hedging sets constructed to hedge the long position in the option. Then it is obvious that the current hedging set is a union of E and F: $G = E \cup F$.

When taking a union of hedging sets E and F the resulting set G may be not convex. It does not change the algorithm presented in section 2 except the implementation of step 3 of stage 3 is a rather more complicated.

Next, the superhedging problem for American options is stated formally. The equations for the top limit are:

(25) $$P(T,i) \cdot a(T,i) + b(T,i) \geq C(T,i) \quad \text{for} \quad i = \overline{1, 2^{T-1}},$$

(26) $$\begin{cases} P(t+1, j(i))\cdot a(t,i) + (1+r)\cdot b(t,i) \geq \\ \quad \geq P(t+1, j(i))\cdot a(t+1, j(i)) + b(t+1, j(i)) + \\ \quad + k\cdot P(t+1, j(i))\cdot | a(t+1, j(i)) - a(t,i)| \\ P(t,i)\cdot a(t,i) + b(t,i) \geq C(t,i) \end{cases}$$

for $t = \overline{1, T-1}$, $i = \overline{1, 2^{t-1}}$ and $j(i) \in \{2i-1, 2i\}$,

(27) $$\overline{C} = \min_{\{a,b\}}[P(1,1)\cdot a(1,1) + b(1,1)],$$

where $\overline{C}$ denotes the top limit of an interval of arbitrage-free prices for an American option.

The equations for the bottom limit are:

(28) $$P(T,i)\cdot a(T,i) + b(T,i) \geq -C(T,i) \quad \text{for} \quad i = \overline{1, 2^{T-1}},$$

(29) $$\left[\begin{array}{l} \begin{cases} P(t+1, 2i-1)\cdot a(t,i) + (1+r)\cdot b(t,i) \geq \\ \quad \geq P(t+1, 2i-1)\cdot a(t+1, 2i-1) + b(t+1, 2i-1) + \\ \quad + k\cdot P(t+1, 2i-1)\cdot | a(t+1, 2i-1) - a(t,i)| \\ \\ P(t+1, 2i)\cdot a(t,i) + (1+r)\cdot b(t,i) \geq \\ \quad \geq P(t+1, 2i)\cdot a(t+1, 2i) + b(t+1, 2i) + \\ \quad + k\cdot P(t+1, 2i)\cdot | a(t+1, 2i) - a(t,i)| \end{cases} \\ P(t,i)\cdot a(t,i) + b(t,i) \geq -C(t,i) \end{array}\right.$$

for $t = \overline{1, T-1}$, $i = \overline{1, 2^{t-1}}$,

(30) $$\underline{C} = -\min_{\{a,b\}}[P(1,1)\cdot a(1,1) + b(1,1)],$$

where $\underline{C}$ denotes the bottom limit of an interval of arbitrage-free prices for an American option.

Respectively equations (14)-(16), i.e. the equations deriving the top limit in the way proposed in this article, change into:

(31) $$X(T,i) = \{(a,b) : P(T,i)\cdot a + b \geq C(T,i)\} \quad \text{for} \quad i = \overline{1, T},$$

(32) $$X(t,i) = \{(u,v) : P(t,i)\cdot u + v \geq P(t,i)\cdot a + b + k\cdot P(t,i)|u-a|,$$
$$(a, (1+r)b) \in X(t+1,i) \cap X(t+1,i+1)\} \cap \{(u,v) : P(t,i)\cdot u + v \geq C(t,i)\}$$

for $t = \overline{1, T-1}$ and $i = \overline{1, t}$,

(33) $$\overline{C} = \min_{(a,b)\in X(1,1)}(P(1,1)\cdot a + b).$$

And equations (17)-(19) transform into:

(34) $$X(T,i)=\{(a,b): P(T,i)\cdot a+b\geq -C(T,i)\} \quad \text{for} \quad i=\overline{1,T},$$

(35) $$X(t,i)=\Big\{(u,v): P(t,i)\cdot u+v\geq P(t,i)\cdot a+b+k\cdot P(t,i)\,|\,u-a\,|,$$
$$(a,(1+r)b)\in X(t+1,i)\cap X(t+1,i+1)\Big\}\cup\{(u,v): P(t,i)\cdot u+v\geq -C(t,i)\}$$
$$\text{for} \quad t=\overline{1,T-1} \quad \text{and} \quad i=\overline{1,t},$$

(36) $$\underline{C}=-\min_{(a,b)\in X(1,1)}(P(1,1)\cdot a+b).$$

As before:

(37) $$\alpha(t,i)=\frac{(1+r)P(t,i)-P(t+1,i)}{P(t+1,i+1)-P(t+1,i)}.$$

Then if the $0\leq\alpha(t,i)\leq 1$ condition holds for every $t=\overline{1,T-1}$ and for every $i=\overline{1,t}$ it is true that $\overline{C}\geq\underline{C}$, as also with the case of European options. It is proved by lemma 3 (appendix 1).

Table 3. Pricing American Options.

The first interval is an interval of arbitrage-free prices for an American option. The second one is an interval of arbitrage-free prices for a European option. T stands for the number of periods; k is the rate of the trade commission levied on the underlying. $P(1,1)=100$ – the current price of the underlying; $K=110$ – the strike price; $\sigma=0.3$ – the volatility of the underlying; $R=0.1$ – the annual interest rate; **T** = 1 – time to maturity.

Call options

T	$k=0.00$	$k=0.01$	$k=0.02$	$k=0.03$
16	12.248 - 12.248 12.248 - 12.248	10.681 - 13.665 10.649 - 13.665	8.829 - 14.952 8.768 – 14.952	6.484 - 16.142 6.370 - 16.142
32	12.085 - 12.085 12.085 - 12.085	9.658 - 14.106 9.641 - 14.106	6.459 - 15.868 6.281 – 15.868	3.527 - 17.452 3.010 - 17.452
64	12.148 - 12.148 12.148 - 12.148	8.495 - 14.930 8.457 - 14.930	3.009 - 17.256 2.875 – 17.256	2.138 - 19.293 1.969 - 19.293
128	12.133 - 12.133 12.133 - 12.133	6.235 - 15.936 6.210 - 15.936	1.734 - 18.961 1.510 – 18.961	1.389 - 21.534 0.950 - 21.534

Put options

T	$k=0.00$	$k=0.01$	$k=0.02$	$k=0.03$
16	13.901 - 13.901 11.780 - 11.780	12.553 - 15.188 10.181 - 13.197	11.376 - 16.411 8.300 – 14.485	10.294 - 17.524 5.902 - 15.674
32	13.838 - 13.838 11.617 - 11.617	11.877 - 15.684 9.173 - 13.638	10.257 - 17.347 5.813 – 15.400	9.729 - 18.865 2.543 - 16.984

64	13.869 - 13.869 11.680 - 11.680	10.956 - 16.442 7.989 - 14.462	9.828 - 18.667 2.407 – 16.788	9.828 - 20.647 1.501 - 18.826
128	13.859 - 13.859 11.665 - 11.665	10.063 - 17.397 5.742 - 15.469	9.914 - 20.320 1.042 – 18.493	9.914 - 22.848 0.482 - 21.066

The following situations can be found among the calculation results:

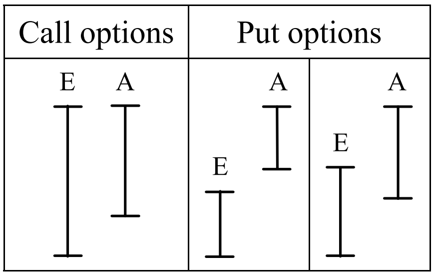

Fig. 6. A schematic illustration of relationships between the intervals of arbitrage-free prices for European and American options.

It can be proved that under the absence of transaction costs the arbitrage-free prices of correspondent American and European call options are the same. At presence of transaction costs the calculations show the same top limits of intervals of arbitrage-free prices for these options. As for the put options the calculations show the limits of their intervals differ from each other and the intervals itself may either intersect or not. In general the correspondent limits are always equal or higher for American options in comparison with European options. It should be that way as the American option gives at least the same possibilities to its holder as the European option does.

It was mentioned earlier that an interval of arbitrage-free prices of a European option expands regarding the number of periods (T). As it is shown in table 3 this observation also suits American options.

Conclusions

The superhedging approach is one of the approaches of pricing derivatives with transaction costs. This work has introduced the new algorithm of the superhedging approach implementation. The algorithm is based on the hedging set idea. The article defined a hedging set, roughly speaking, is a set built by portfolios accompanied by trading strategies with consumption that guarantee ability to pay the considered obligations at any outcome. This algorithm allows an efficient solution to the superhedging problem and has a clear geometrical interpretation.

The usage of the algorithm for pricing European and American options under the presence of proportional transaction costs was studied and the appropriate numerical examples were given. Wherever it was possible the numerical examples included the comparison with other relevant algorithms.

The author assumes application of hedging sets to the superhedging approach promising as it allows using trees of any type to model prices of an underlying. Also multidimensional hedging sets may allow pricing and hedging of derivatives that depend on several random factors.

APPENDICES

Appendix 1. Miscellaneous Proofs

Equations (14)-(16) give the top limit of an interval of arbitrage-free prices of a European option. Let us denote the hedging sets used in this equations by $\widehat{X}(t,i)$. Equations (17)-(19) give the bottom limit. Let us denote the correspondent hedging sets by $\breve{X}(t,i)$.

Let

(38) $$\widehat{C}(t,i) = \min\{P(t,i)\cdot a + b \mid (a,b)\in \widehat{X}(t,i)\},$$

(39) $$\breve{C}(t,i) = -\min\{P(t,i)\cdot a + b \mid (a,b)\in \breve{X}(t,i)\}.$$

It should be noted that

(40) $$\alpha(t,i) = \frac{(1+r)P(t,i) - P(t+1,i)}{P(t+1,i+1) - P(t+1,i)}.$$

Lemma 1. *Let* $0 \le \alpha(t,i) \le 1$ *for some* t *and* i. *Then for these* t *and* i *the following is true*

$$\widehat{C}(t,i) \ge \frac{1}{1+r}\left(\alpha(t,i)\cdot \widehat{C}(t+1,i+1) + (1-\alpha(t,i))\cdot \widehat{C}(t+1,i)\right),$$

$$\breve{C}(t,i) \le \frac{1}{1+r}\left(\alpha(t,i)\cdot \breve{C}(t+1,i+1) + (1-\alpha(t,i))\cdot \breve{C}(t+1,i)\right).$$

Proof. The first statement comes from:

(41) $$\begin{aligned}\widehat{C}(t,i) &= \min\left\{ P(t,i)\cdot u + v \,\middle|\, (u,v)\in \widehat{X}(t,i) \right\} = \\ &= \min\left\{ P(t,i)\cdot u + v \,\middle|\, P(t,i)\cdot u + v \ge P(t,i)\cdot a + b + k\cdot P(t,i)\,|u-a|,\right. \\ &\qquad \left. (a,(1+r)b)\in \widehat{X}(t+1,i+1)\cap \widehat{X}(t+1,i) \right\} \ge \\ &\ge \min\left\{ P(t,i)\cdot u + v \,\middle|\, P(t,i)\cdot u + v \ge P(t,i)\cdot a + b,\right. \\ &\qquad \left. (a,(1+r)b)\in \widehat{X}(t+1,i+1)\cap \widehat{X}(t+1,i) \right\} = \\ &= \min\left\{ P(t,i)\cdot a + b \,\middle|\, (a,(1+r)b)\in \widehat{X}(t+1,i+1)\cap \widehat{X}(t+1,i) \right\} = \\ &= \frac{1}{1+r}\min\left\{ (1+r)P(t,i)\cdot a + \widetilde{b} \,\middle|\, (a,\widetilde{b})\in \widehat{X}(t+1,i+1)\cap \widehat{X}(t+1,i) \right\} =\end{aligned}$$

$$=\frac{1}{1+r}\min\{\,\alpha(t,i)(P(t+1,i+1)\cdot a+\widetilde{b})+(1-\alpha(t,i))(P(t+1,i)\cdot a+\widetilde{b})\;|$$

$$(a,\widetilde{b})\in\widehat{X}(t+1,i+1)\cap\widehat{X}(t+1,i)\,\}\geq$$

$$\geq\frac{1}{1+r}\Big(\,\min\{\,\alpha(t,i)(P(t+1,i+1)\cdot a+\widetilde{b})\,|\,(a,\widetilde{b})\in\widehat{X}(t+1,i+1)\cap\widehat{X}(t+1,i)\,\}+$$

$$+\min\{\,(1-\alpha(t,i))(P(t+1,i)\cdot a+\widetilde{b})\,|\,(a,\widetilde{b})\in\widehat{X}(t+1,i+1)\cap\widehat{X}(t+1,i)\,\}\,\Big)\geq$$

$$\geq\frac{1}{1+r}\Big(\,\min\{\,\alpha(t,i)(P(t+1,i+1)\cdot a+\widetilde{b})\,|\,(a,\widetilde{b})\in\widehat{X}(t+1,i+1)\,\}+$$

$$+\min\{\,(1-\alpha(t,i))(P(t+1,i)\cdot a+\widetilde{b})\,|\,(a,\widetilde{b})\in\widehat{X}(t+1,i)\,\}\,\Big)=$$

$$=\frac{1}{1+r}\Big(\,\alpha(t,i)\min\{\,P(t+1,i+1)\cdot a+\widetilde{b}\,|\,(a,\widetilde{b})\in\widehat{X}(t+1,i+1)\,\}+$$

$$+(1-\alpha(t,i))\min\{\,P(t+1,i)\cdot a+\widetilde{b}\,|\,(a,\widetilde{b})\in\widehat{X}(t+1,i)\,\}\,\Big)=$$

$$=\frac{1}{1+r}\Big(\,\alpha(t,i)\cdot\widehat{C}(t+1,i+1)+(1-\alpha(t,i))\cdot\widehat{C}(t+1,i)\Big).$$

The second statement is proved the same way.

□

Lemma 2. *Let* $0\leq\alpha(t,i)\leq 1$ *for every* $t=\overline{1,T-1}$ *and for every* $i=\overline{1,t}$. *Then* $\widehat{C}(1,1)\geq\breve{C}(1,1)$.

Proof. From the definition of hedging sets $\widehat{X}(T,i)$ and $\breve{X}(t,i)$ (equations (14) and (17), respectively) it follows that $\widehat{C}(T,i)\geq\breve{C}(T,i)$ for every $i=\overline{1,T}$. Let us suppose that $\widehat{C}(t+1,i)\geq\breve{C}(t+1,i)$ for every $i=\overline{1,t+1}$. Then Lemma 1 gives:

(42) $$\widehat{C}(t,i)\geq\frac{1}{1+r}\Big(\,\alpha(t,i)\cdot\widehat{C}(t+1,i+1)+(1-\alpha(t,i))\cdot\widehat{C}(t+1,i)\Big)\geq$$

$$\geq\frac{1}{1+r}\Big(\,\alpha(t,i)\cdot\breve{C}(t+1,i+1)+(1-\alpha(t,i))\cdot\breve{C}(t+1,i)\Big)\geq\breve{C}(t,i).$$

And from induction it follows that $\widehat{C}(1,1)\geq\breve{C}(1,1)$.

□

Equations (31)-(33) and (34)-(36) give the top and the bottom limits of an interval of arbitrage-free prices of an American option respectively. Let us denote the correspondent hedging sets by $\widehat{X}_a$ and $\breve{X}_a$.

Let

(43) $$\widehat{C}_a(t,i)=\min\{P(t,i)\cdot a+b\,|\,(a,b)\in\widehat{X}_a(t,i)\},$$

(44) $$\breve{C}_a(t,i) = -\min\{P(t,i)\cdot a + b \mid (a,b) \in \breve{X}_a(t,i)\}.$$

Lemma 3. *Let* $0 \le \alpha(t,i) \le 1$ *for every* $t = \overline{1, T-1}$ *and for every* $i = \overline{1,t}$. *Then* $\widehat{C}_a(1,1) \ge \breve{C}_a(1,1)$.

Proof. Let

(45) $$\widehat{E}(t,i) = \big\{ (u,v) : P(t,i)\cdot u + v \ge P(t,i)\cdot a + b + k\cdot P(t,i)\,|u-a|, \\ (a,(1+r)b) \in \widehat{X}_a(t+1,i) \cap \widehat{X}_a(t+1,i+1) \big\},$$

(46) $$\breve{E}(t,i) = \big\{ (u,v) : P(t,i)\cdot u + v \ge P(t,i)\cdot a + b + k\cdot P(t,i)\,|u-a|, \\ (a,(1+r)b) \in \breve{X}_a(t+1,i) \cap \breve{X}_a(t+1,i+1) \big\}.$$

Let us also define

(47) $$\widehat{G}(t,i) = \{(a,b) : P(t,i)\cdot a + b \ge C(t,i)\},$$

(48) $$\breve{G}(t,i) = \{(a,b) : P(t,i)\cdot a + b \ge -C(t,i)\},$$

where $C(t,i)$ stands for the amount of received money if the considered American option is exercised at moment t and state i. New notations redraw equations (32), (35) in the following way:

(49) $$\widehat{X}_a(t,i) = \widehat{E}(t,i) \cap \widehat{G}(t,i),$$

(50) $$\breve{X}_a(t,i) = \breve{E}(t,i) \cup \breve{G}(t,i).$$

Let

(51) $$\widehat{Z}(t,i) = \min\{P(t,i)\cdot a + b \mid (a,b) \in \widehat{E}(t,i)\},$$

(52) $$\breve{Z}(t,i) = -\min\{P(t,i)\cdot a + b \mid (a,b) \in \breve{E}(t,i)\}.$$

It is obvious that $\widehat{C}_a(T,i) \ge \breve{C}_a(T,i)$ for every $i = \overline{1,T}$. Let us suppose $\widehat{C}_a(t+1,i) \ge \breve{C}_a(t+1,i)$ for every $i = \overline{1,t+1}$. From the same reasoning as in lemmas 1 and 2 it is:

(53) $$\widehat{Z}(t,i) \ge \breve{Z}(t,i) \quad \text{for} \quad i = \overline{1,t}.$$

Next

(54) $$\widehat{C}_a(t,i) = \min\{P(t,i)\cdot a + b \mid (a,b) \in \widehat{X}_a(t,i)\} = \\ = \min\{P(t,i)\cdot a + b \mid (a,b) \in \widehat{E}(t,i) \cap \widehat{G}(t,i)\} \ge \\ \ge \max\big(\min\{P(t,i)\cdot a + b \mid (a,b) \in \widehat{E}(t,i)\},\ \min\{P(t,i)\cdot a + b \mid (a,b) \in \widehat{G}(t,i)\}\big) = \\ = \max(\widehat{Z}(t,i), C(t,i)).$$

Also

$$(55)\qquad \breve{C}_a(t,i) = -\min\{P(t,i)\cdot a + b \mid (a,b)\in \breve{X}_a(t,i)\} =$$
$$= -\min\{P(t,i)\cdot a + b \mid (a,b)\in \breve{E}(t,i)\cup \breve{G}(t,i)\} =$$
$$= -\min\big(\min\{P(t,i)\cdot a + b \mid (a,b)\in \breve{E}(t,i)\},\ \min\{P(t,i)\cdot a + b \mid (a,b)\in \breve{G}(t,i)\}\big) =$$
$$= \max\big(-\min\{P(t,i)\cdot a + b \mid (a,b)\in \breve{E}(t,i)\},\ -\min\{P(t,i)\cdot a + b \mid (a,b)\in \breve{G}(t,i)\}\big) =$$
$$= \max(\breve{Z}(t,i), C(t,i)).$$

It gives

$$(56)\qquad \widehat{C}_a(t,i) \ge \max(\widehat{Z}(t,i), C(t,i)) \ge \max(\breve{Z}(t,i), C(t,i)) = \breve{C}_a(t,i).$$

Finally, induction gives the required statement.

□

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