

Drugs, Guns, and Targeted Competition

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Abstract

We consider a dynamic competition game among three players, where each player can vary the extent of his competition on a per-rival basis. We call such competition targeted. We show that if the players are myopic, the weaker players eventually lose the game to their strongest rival. If instead the players are sufficiently far-sighted, then all three players converge in their power and stay in the game. We develop our model in application to drug wars, but the approach of targeted competition can be applied to competition between firms or political parties, or to warfare.

Key Words: targeted competition, dynamic oligopoly, differential games, drug wars.

JEL Classification: C73, D43.

1 Introduction

Competition lies at the heart of economics and has been studied extensively. However, there is a class of competition mechanisms which abounds in practice but which, to the best of our knowledge, has not been studied specifically in the literature. Those are the mechanisms that provide a competitor with an ability to target his rivals on individual bases. We call such mechanisms targeted competition. In differentiated product markets firms can develop products that are closer in certain characteristics to that of particular competitors. Multinational corporations¹ can invest relatively more in markets shared with particular rivals. Firms, especially in the U.S., employ comparative advertisement—the practice of running ads that directly compare one’s

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¹See, e.g., surveys by Lancaster (1990); Gabszewicz and Thisse (1992); Bailey and Friedlaender (1982).

products to those of one's rivals.² Unethical practices, e.g. launching fabricated lawsuits, provide additional ways to harm particular competitors. Targeted competition is not restricted to the field of industrial organization. For instance, political parties and politicians compete through their support for specific programs, and at times governments protect national industries through trade barriers. Finally, warfare, whether locally among crime groups or globally among nations, remains the ultimate example of targeted competition.³

Targeted competition poses a strategic consideration that does not arise in non-targeted competition. An individual player (a firm, a political party, a crime group) can influence the balance of power among his rivals by choosing whom to compete against, and this in turn determines how much this player wins or loses in competing with his rivals in the periods to come. In particular, one may intuitively expect the weaker players to direct more resources towards fighting the strongest player than to fighting each other. Otherwise the strongest player stands a good chance of forcing the weaker players out of the game as time goes by. However, such intuition demands further exploration: if one weaker player fights more against his strongest rival, the remaining weak players have free-riding incentives that can undermine the fight against the strongest player.

Any model of targeted competition should have the following two properties: 1) there should be three or more players—otherwise, the competition cannot be targeted; and 2) the analysis should be dynamic, as the aforementioned strategic consideration can only be studied in a dynamic setting. The closest matching strand of the literature then is that of dynamic oligopoly models. Though many scenarios of dynamic competition have been studied—inventories (Kirman and Sobel, 1974), sticky prices (Fershtman and Kamien, 1987), evolution of sales (Dockner and Jørgensen, 1988), capacity adjustment costs (Reynolds, 1991), varying profit opportunities (Ericson and Pakes, 1995), capital accumulation (Cellini and Lambertini, 1998), collusive behaviour (Fershtman and Pakes, 2000), etc.—targeted competition has not been a part of the analysis.

In order to develop a formal discussion of targeted competition, we focus on a particular scenario: competition between drug cartels. In our view, this scenario best highlights the nature of targeted competition. On one hand, drug cartels ultimately aim at profits received from drug sales. On the other hand, these organizations do not shy away from re-recruiting or simply eliminating their opponents' members. The latter was the case during the drug wars among Colombian drug cartels in the 80's and 90's, and is still the case in the ongoing Mexican drug war.

Each drug cartel in our model is characterised by its power: the num-

²See, e.g., Barigozzi and Peitz (2007).

³See Blattman and Miguel (2010) for a review of the literature on civil wars.

ber of men (or women)⁴ that the cartel commands. Cartel’s manpower is employed for drug sales as well as for combating cartel’s rivals. Drug sales yield profits, whereas fighting cartel’s rivals shift the balance of power. We first show that myopic cartels prefer to fight more with their weakest opponent. Consequently, the strongest cartel increases his power and eventually out-competes his weaker rivals. Inversely, we also show that if cartels are non-myopic and do not discount future profits too much, then the weaker cartels concentrate more on fighting their strongest opponent (provided that no cartel is initially too strong). Consequently, the strongest cartel becomes weaker over time, and all the cartels reach a common power level and their drug war persists.

To the extent to which our model could encompass other cases of competition, we argue the following. If competition in a certain market is targeted and if the competitors are forward-looking, then competition is sustainable. On the other hand, if there is no way to target particular rivals, then the market becomes a monopoly.

It is tempting to view the efforts of the weaker cartels to fight against their strongest rival as a form of tacit collusion. The idea at play, however, is conceptually different. Collusive behaviour in repeated games is sustained by the credible threat that other players will punish the one who deviates from the equilibrium. In our game, the equilibrium is a Markov perfect equilibrium. Hence, the strategies do not depend upon past actions, and there are no strategies involving retrospective punishment. In this scenario, it is the dynamic structure of the game that pushes the weaker cartels to fight together for the common cause. If they prefer to fight each other for the sake of immediate gains, then the power of the strongest cartel will eventually grow to the point at which it can out-compete its rivals. To avoid this loss of future profits the weaker cartels fight more against their strongest rival, and so their behaviour resembles tacit collusion.

There are two related games that have been studied in the literature: colonel Blotto games (see, e.g., Roberson, 2006; Hart, 2008; Arad and Rubinstein, 2011) and truel games (Kilgour, 1971).

A colonel Blotto game is a game between two players who share several battlefields. Each player divides his army between battlefields, each battlefield is won by the larger force, and the player who wins more battlefields wins the game. The main differences with our game are as follows: 1) there are three players in our game and 2) our game is dynamic: the winner is not determined at once; rather, the winner of each round becomes stronger and the game continues. In principal, the game of targeted competition that we study can be viewed as a game with three players and three battlefields in which each pair of players shares a battlefield and in which there is no battlefield that is shared by all of the players. Under such conditions, our game

⁴E.g., sicarias (female assassins).

is similar to colonel Blotto games because the players are able to choose how to split their power against their opponents.

A truel game is an extension of a duel game. There are three players, each with a gun. Each round each player chooses whom to shoot, and his chance of killing his opponent depends upon his skill; if two or more players are still alive, the game continues. As in our game, each player chooses his opponent, and killing a certain player influences the killer's chance of survival in the rounds to come. The main differences are that 1) in our game the payoff from the game is the discounted sum of the payoffs in each round, so that each round is valuable, whereas in a truel game the payoff is 1 if the player survives and 0 otherwise; 2) in our game, if the player is "shot", he does not die at once but rather becomes relatively weaker; and 3) in a truel game a player chooses to fight either one opponent or the other, whereas in our game a player chooses *how much* to fight one opponent and *how much* to fight the other (a continuous choice).

So, our game has structural similarities to those of colonel Blotto and truel games, but we find that the named differences make our modelling approach more suitable for the study of drug wars, and, potentially, for the study of other cases of targeted competition.

The rest of the paper is organised as follows. The next section presents a model of targeted competition, which is based on the scenario of drug wars between drug cartels. Section three discusses the implications of the model for the case of myopic cartels and for the case of forward-looking cartels. The last section concludes. All formal proofs are located in the appendix.

2 Setup

There are three drug cartels labelled 1, 2, and 3, which are involved in a lasting armed conflict over their "export" markets. Each cartel i at time $t \in [0, \infty)$ is characterised by its manpower $x_i(t)$, which the cartel can employ for competition against its rivals. For brevity, we refer to x_i as the *power* of cartel i .

Denote $x = (x_1, x_2, x_3)$. At any time t , the powers of the cartels, $x(t)$, are common knowledge. The initial state $x(0)$ is normalised so that $\sum_i x_i(0) = 1$ (later on we will see that $\sum_i x_i(t) = 1$ for any t). We also assume that no cartel is too strong to start with. Formally, $x(0) \in X$, where

$$X = \left\{ x \in \mathbb{R}^3 \left| \sum_i x_i = 1, x_i < \frac{2}{5} \forall i \right. \right\}.$$

The reason for the restriction $x_i(0) < \frac{2}{5}$ is a technical one. Under this restriction, the best responses (which we are to analyse later on) are inner solutions, and the whole problem is analytically tractable. If one considers a more natural restriction that $x_i(0) < \frac{1}{2}$, then one needs to derive a numerical

solution to a system of differential equations. We avoid the difficulty of solving the problem numerically by considering a smaller region for x .

There are three “export” markets for drugs, and each market is served by a different pair of competing cartels. Each cartel can freely allocate its power among its markets, thus targeting particular rivals. Let y_{ij} denote the amount of power that cartel i allocates to the market shared with cartel j . Further, let $y_1 = (y_{12}, y_{13})$, $y_2 = (y_{21}, y_{23})$, $y_3 = (y_{31}, y_{32})$ and $y = (y_1, y_2, y_3)$.

In the following analysis we focus on Markov strategies, under which the choices of y_i by each cartel depend only on the current state x and not on the past actions of the cartels. We choose Markov strategies because of the objective of the paper—to study whether forward-looking behaviour can produce collusive outcomes, but without the usual means of sustaining collusion (such as trigger strategies). Moreover, it is appealing to consider Markov strategies for several other reasons. First, an equilibrium in Markov strategies is also an equilibrium in a game with non-Markov strategies. Second, suppose that a game involving general strategies has multiple equilibria, one of which is a Markov equilibrium. One way to select an equilibrium is to explore whether there is a focal point (Schelling, 1960). If simplicity makes a focal point, then the Markov equilibrium is selected. There are also other reasons, both theoretical and practical, for opting for Markov strategies; see the introduction to Maskin and Tirole (2001).

Each cartel uses all of its power against its opponents and the amount of power used cannot be negative. Given Markov strategies $y(x)$, we have

$$y_i(x) \in Y_i(x), \quad (1)$$

$$Y_i(x) = \left\{ y_i \mid y_{ij} \geq 0, \sum_j y_{ij} = x_i \right\}.$$

Each market is identical and has the following inverse demand for drugs:

$$p_{ij} = a - bq_{ij},$$

where $b > 0$, $a \geq 3b$, and i, j stand for the participating cartels. On each market the cartels engage in a Cournot competition. The amount of drugs that cartel i can supply to market ij is strictly proportional to its presence on the market, i.e. to y_{ij} . Hence, $q_{ij} = y_{ij} + y_{ji}$ and the instantaneous profits of cartel i from market ij are given by

$$\varphi(y_{ij}, y_{ji}) = (a - b(y_{ij} + y_{ji}))y_{ij},$$

Let $\pi_i(y)$ denote the total instantaneous profits of cartel i . We have

$$\pi_i(y) = \sum_{j \neq i} \varphi(y_{ij}, y_{ji}).$$

Power is not a factor in the instantaneous profit function per se. However, becoming more powerful will yield higher profits because one can use more power to compete against one's rivals, thus improving the outcomes of future rounds of competition.

Over time each cartel recruits new members, whose number is proportional to the current manpower of the cartel. On the other hand, all three cartels are engaged in an armed conflict against each other on the markets they share. The corresponding losses of a cartel are proportional to the amount of manpower allocated against it by its rivals. Formally,

$$\dot{x}_i(t) = k \left(x_i(t) - \sum_{j \neq i} y_{ji}(x(t)) \right), \quad (2)$$

where $k > 0$ is the proportionality coefficient. For simplicity, k is taken to be the same for cartel's growth and losses. Consequently, from $\sum_i x_i(0) = 1$ and from (2) it follows that $\sum_i x_i(t) = 1$ for all t .

If $x(t)$ reaches the boundary of X , the game ends. T denotes the ending time. Formally,

$$T = \inf\{t \geq 0 \mid x(t) \notin X\}.$$

If the game never ends, we write $T = \infty$.

If the game ends, each cartel i receives a terminal profit S_i , the strongest cartel wins, and the weaker cartels lose:

$$S_i(x) = \begin{cases} M & \text{if } x_i > x_j \ \forall j \neq i, \\ 0 & \text{otherwise,} \end{cases}$$

where $M > 0$.⁵

The rationale for ending the game if the boundary of X is approached is as follows. If one of the cartels becomes sufficiently strong, it is reasonable to expect that this cartel eventually out-competes its rivals. To simplify the game, we stop it at this moment and assign a strictly positive profit M to the strongest cartel and a zero profit to the weaker cartels.

The profit for the whole game is the discounted stream of the instantaneous profits plus the discounted terminal profit, so the profit for cartel i is

$$U_i = \int_0^T e^{-rt} \pi_i(y(x(t))) dt + e^{-rT} S_i(x(T)), \quad (3)$$

where r is an instantaneous discount rate. Alternatively, r can be viewed as a hazard rate for cartels' leaders. As long as a new leader is elected whenever the previous leader is assassinated (or otherwise quits the game), our specification continues to hold.

⁵If the game ends and two cartels are equally strong, they both lose. This assumption is made for the sake of simplicity and does not change the results.

Thus, our setup is a differential game with simultaneous play (see Dockner et al., 2000) and we restrict our attention to Markov strategies. The strategies are functions $y(x)$ satisfying (1), the state variables x evolve according to (2) and the objective functions are given by (3).

Alternatively, a repeated game with intervals of length Δt can be set up in a similar fashion. A solution to the repeated game, which can be obtained using the one-stage deviation principle (Fudenberg and Tirole, 1991, sec. 4.2), yields our solution as $\Delta t \rightarrow 0$. However, the analysis of the repeated game is more tedious than the analysis of its differential counterpart. Thus, we have opted for a differential game.

3 Analysis

We consider two cases: a case with myopic cartels and a general case. In both cases, we look for Markov perfect equilibria (MPE) and analyse the resulting equilibrium dynamics.

The cartels are myopic if they focus on current profits. So, for a myopic cartel i the profit of the game at time t is

$$U_i(t) = \pi_i(y(x(t))).$$

The dynamics of the myopic case are summarised in the following proposition (we limit our attention to a general initial state in which one of the cartels is strictly stronger than the rest).

Proposition 1. *Suppose, without loss of generality, that $x_1(0) > x_2(0)$, $x_1(0) > x_3(0)$. Then there exists a unique MPE. This MPE is defined by*

$$y_{ij}(x) = \frac{x_i}{2} + \frac{x_k - x_j}{10}$$

and its dynamics are given by

$$\dot{x}_i(t) = \frac{9k}{5} \left(x_i(t) - \frac{1}{3} \right).$$

In this equilibrium the game ends and the strongest cartel wins: $T < \infty$, $x_1(T) > x_2(T)$, $x_1(T) > x_3(T)$.

So, myopic cartels fight more against each other than against their strongest rival. This result occurs because investing in a market shared with a weaker opponent yields higher immediate profits. Consequently, the strongest cartel wins.

There is a small distance between our assumptions and Proposition 1. However, Proposition 1 serves as a benchmark for our setup. We are interested to know whether long term strategic considerations induce the weaker

cartels to balance the power of their strongest rival. If the model had been specified in such a way that even myopic cartels balanced their strongest opponent, then our question would have been ill-posed. As it stands now, the question is well-posed and we proceed with its discussion.

When the cartels value their future profits highly enough (i.e., when r is sufficiently small), the weaker cartels have incentives to fight their strongest opponent more so as to balance his future power and thus earn higher profits in the long run. However, this intuition is incomplete. If one of the weaker cartels spends its power to balance the strongest cartel, then the other weak cartel has incentives to free-ride, and to fight his weaker rival rather than his stronger rival. These free-riding incentives might preclude existence of an equilibrium with converging power levels. The following proposition resolves this ambiguity.

Proposition 2. *If $r < \frac{4k}{3}$, then there exists a unique MPE with linear symmetric strategies such that $x(t) \rightarrow \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ as $t \rightarrow \infty$. If $r \geq \frac{4k}{3}$, then no such equilibria exist.⁶ The equilibrium in question is defined by*

$$y_{ij}(x) = \frac{x_i}{2} + \frac{c(x_k - x_j)}{2},$$

where

$$c = \frac{1}{18} \left(5\frac{r}{k} - 14 - \sqrt{25\left(\frac{r}{k}\right)^2 - 176\frac{r}{k} + 304} \right),$$

and the equilibrium dynamics are given by

$$\dot{x}_i(t) = \frac{3k(c+1)}{2} \left(x_i(t) - \frac{1}{3} \right).$$

For visual purposes, coefficient c can be viewed as a linear function of r/k , because $c \approx 9/16 \cdot r/k - 7/4$, and the error is no more than 0.0036 for $r/k \in (0, 4/3)$.

So, for a sufficiently small r there is an MPE in which the weaker cartels balance the power of the strongest cartel. In this equilibrium each cartel allocates more power against his stronger rival (as $c < -1$). Further, the strongest cartel becomes weaker over time, while the weaker cartels become stronger. Consequently, the power levels of all of the cartels converge. Finally, if the cartels value their future profits more, then the speed of convergence is faster (c is more negative for smaller values of r).

Let us now consider an interpretation, in which r stands for the hazard rate associated with leading a cartel. As we noted earlier, our model specification does not change as long as a new leader is elected whenever the

⁶As the proposition says, we limit our attention to equilibrium strategies that are linear and symmetric, and that lead the cartels towards the centre. However, we do not restrict non-equilibrium strategies in any way: i.e., we consider all possible deviations from equilibrium strategies.

previous leader gets assassinated. Then, if we restrict our attention to MPE in linear symmetric strategies, Proposition 2 gives an empirical prediction: a drug war is more likely to persist for smaller hazard rates. We put this prediction in probability terms as k is likely to be unobservable by a foreign party. This prediction would have been trivial if an assassination of a leader had resulted in a dissolution of his cartel. Our prediction is more subtle. We assume that cartels can outlive their leaders. But, if the hazard rate is high, the weaker cartels are more likely to fight each other, thus the strongest cartel is more likely to win and to monopolize the markets. Inversely, with a low hazard rates the weaker cartels are more likely to balance the strongest cartel, in which case their drug war persists.

4 Concluding Remarks

Stackelberg (1952) has argued that a duopoly can never achieve equilibrium in price/quantity-setting strategies. Moreover, the actors in a duopoly will fight for leadership, and consequently, either one of them will become much stronger in economic terms, or they will find it beneficial to collude.

“Duopoly is an unstable market form not only in the sense that price is apt to be indeterminate, but much more because it is unlikely to remain as a market form for any length of time. The inherent contradictions in the duopolistic situation press for a solution through the adoption of another market form—monopoly”

We do not intend to suggest that a market of three will attain an equilibrium in terms of either prices or quantities. Such strategic variables may well remain indeterminate. Rather, we consider the relative power levels of the players, in our model—the manpower of drug cartels. We show that if the three cartels are sufficiently forward looking, there exists an equilibrium in which everyone competes more against his stronger rival. Consequently, the cartels converge in their power, and oligopolistic competition becomes sustainable—the system does not devolve into monopoly.⁷

Appendix

Because the nature of the players is irrelevant for the formal proofs, in this appendix we refer to the three cartels as players. In the proofs we denote the best response strategies with \tilde{y} and the equilibrium strategies with \hat{y} .

Proposition 1. *Suppose, without loss of generality, that $x_1(0) > x_2(0)$,*

⁷We do not mean to imply that a sustainable competition is always socially better. One can argue that in our particular exposition about drug cartels a drug monopoly is preferable to a competition of three, because the output in terms of drugs is smaller under monopoly.

$x_1(0) > x_3(0)$. Then there exists a unique MPE. This MPE is defined by

$$y_{ij}(x) = \frac{x_i}{2} + \frac{x_k - x_j}{10}$$

and its dynamics are given by

$$\dot{x}_i(t) = \frac{9k}{5} \left(x_i(t) - \frac{1}{3} \right).$$

In this equilibrium the game ends and the strongest player wins: $T < \infty$, $x_1(T) > x_2(T)$, $x_1(T) > x_3(T)$.

Proof. Maximising $U_i(t)$ in (y_{ij}, y_{ik}) w.r.t. $y_{ij} + y_{ik} = x_i$ provides a unique best response

$$\tilde{y}_{ij}(x) = \frac{x_i}{2} + \frac{y_{ki}(x) - y_{ji}(x)}{4}$$

(while a boundary solution is, in principal, possible, it is straightforward to verify that it is never attained for $x \in X$).

Given the above best response functions we can solve for a unique equilibrium point. We obtain

$$\hat{y}_{ij}(x) = \frac{x_i}{2} + \frac{x_k - x_j}{10}. \quad (4)$$

Because we are considering Markov strategies, (4) constitutes a unique Markov perfect equilibrium.

Plugging (4) into (2) and using $x_1 + x_2 + x_3 = 1$ yields

$$\dot{x}_i(t) = \frac{9k}{5} \left(x_i(t) - \frac{1}{3} \right).$$

Therefore,

$$\dot{x}_1(t) - \dot{x}_i(t) = \frac{9k}{5} (x_1(t) - x_i(t)). \quad (5)$$

X is bounded, and $x_1(0) > x_i(0)$ for $i \in \{2, 3\}$. It then follows from (5) that $x(t)$ reaches the boundary of X at some time T and that $x_1(T) > x_i(T)$ for $i \in \{2, 3\}$. \square

Proposition 2. *If $r < \frac{4k}{3}$, then there exists a unique MPE with linear symmetric strategies such that $x(t) \rightarrow \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ as $t \rightarrow \infty$. If $r \geq \frac{4k}{3}$, then no such equilibria exist. The equilibrium in question is defined by*

$$y_{ij}(x) = \frac{x_i}{2} + \frac{c(x_k - x_j)}{2},$$

where

$$c = \frac{1}{18} \left(5\frac{r}{k} - 14 - \sqrt{25\left(\frac{r}{k}\right)^2 - 176\frac{r}{k} + 304} \right),$$

and the equilibrium dynamics are given by

$$\dot{x}_i(t) = \frac{3k(c+1)}{2} \left(x_i(t) - \frac{1}{3} \right).$$

Proof. An equilibrium strategy \hat{y}_{ij} is linear if

$$\hat{y}_{ij}(x) = \alpha_{ij}x_i + \beta_{ij}x_j + \gamma_{ij}x_k + \delta_{ij}, \quad (6)$$

where α_{ij} , β_{ij} , γ_{ij} and δ_{ij} are arbitrary but constant coefficients.

These strategies are symmetric if 1) $\hat{y}_{ij} = \hat{y}_{ik}$ whenever $x_j = x_k$, and 2) $\hat{y}_{ij} = \hat{y}_{ji}$ whenever $x_i = x_j$. Imposing these symmetry conditions on (6) and using $\sum_i x_i = 1$ and $y_{ij} + y_{ik} = x_i$, yields

$$\alpha_{ij} + \delta_{ij} = \frac{1}{2}, \quad \beta_{ij} + \delta_{ij} = \beta_{ji} + \delta_{ji} = -(\gamma_{ij} + \delta_{ij}).$$

Let $c = 2(\gamma_{ij} + \delta_{ij})$. Then

$$\hat{y}_{ij}(x) = \frac{x_i + c(x_k - x_j)}{2}. \quad (7)$$

From $\sum_i x_i(t) = 1$, from (2) and from (7), it follows that

$$\dot{x}_i(t) = \frac{3k(c+1)}{2} \left(x_i(t) - \frac{1}{3} \right). \quad (8)$$

So, $x_i(t) \rightarrow \frac{1}{3}$ for all i as $t \rightarrow \infty$ if and only if $c < -1$.

To prove the proposition, we need to show that for $r < \frac{4k}{3}$ there exists a unique $c < -1$ such that (7) constitutes an MPE, and that for $r \geq \frac{4k}{3}$ no such c exists.

To investigate when (7) constitutes an MPE, we need to investigate when \hat{y}_i is the best response to \hat{y}_j and \hat{y}_k . So, we fix the strategies of players j and k at \hat{y}_j and \hat{y}_k and consider different strategies of player i . Given the strategies of players j and k , all possible strategies of player i can be divided into two classes: those strategies that never end the game ($T = \infty$)—let this be class \mathcal{A} , and those that eventually do ($T < \infty$)—class \mathcal{B} . We proceed as follows. First, we restrict the strategies of player i to class \mathcal{A} and investigate when the strategy \hat{y}_i , as given by (7), is indeed the best response strategy. Then, we extend our results to $\mathcal{A} \cup \mathcal{B}$.

So, let the strategies of player i be restricted to class \mathcal{A} . Let us compute the value function V of player i if every player follows strategy \hat{y} and if the game starts at $x(0) = x$. Solving (8) gives us

$$x_i(t) = \left(x_i - \frac{1}{3} \right) e^{3k(c+1)/2 \cdot t} + \frac{1}{3}. \quad (9)$$

Let $z_i = x_i - \frac{1}{3}$. As $x_1 + x_2 + x_3 = 1$, we obtain $z_1 + z_2 + z_3 = 0$ and

$$\hat{y}_{ij}(x) = \frac{x_i + c(x_k - x_j)}{2} = \frac{z_i + c(z_k - z_j)}{2} + \frac{1}{6}.$$

Then (using $\sum_i z_i = 0$ where appropriate)

$$\begin{aligned} \pi_i(\hat{y}(z)) &= (a - b(\hat{y}_{ij} + \hat{y}_{ji}))\hat{y}_{ij} + (a - b(\hat{y}_{ik} + \hat{y}_{ki}))\hat{y}_{ik} = \\ &\left(a - b \left(\frac{z_i + c(z_k - z_j)}{2} + \frac{z_j + c(z_k - z_i)}{2} + \frac{1}{3} \right) \right) \cdot \\ &\quad \left(\frac{z_i + c(z_k - z_j)}{2} + \frac{1}{6} \right) + \\ &\left(a - b \left(\frac{z_i + c(z_j - z_k)}{2} + \frac{z_k + c(z_j - z_i)}{2} + \frac{1}{3} \right) \right) \cdot \\ &\quad \left(\frac{z_i + c(z_j - z_k)}{2} + \frac{1}{6} \right) = \\ &\left(a - \frac{b}{3} \right) \left(z_i + \frac{1}{3} \right) - \frac{b(3c-1)}{2} \left(z_k \left(\frac{z_i + c(z_k - z_j)}{2} + \frac{1}{6} \right) + \right. \\ &\quad \left. z_j \left(\frac{z_i + c(z_j - z_k)}{2} + \frac{1}{6} \right) \right) = \\ &\frac{b(3c-1)}{4} z_i^2 + \frac{12a + b(3c-5)}{12} z_i + \frac{3a-b}{9} - \frac{bc(3c-1)}{4} (z_k - z_j)^2. \end{aligned}$$

Let $m = 3k(c+1)/2$, then (9) gives $z_i(t) = z_i e^{mt}$. So,

$$\begin{aligned} V_i(z) &= \int_0^\infty e^{-rt} \pi_i(\hat{y}(z(t))) dt = \\ &\int_0^\infty e^{-rt} \left(\frac{b(3c-1)}{4} (z_i e^{mt})^2 + \frac{12a + b(3c-5)}{12} z_i e^{mt} + \right. \\ &\quad \left. \frac{3a-b}{9} - \frac{bc(3c-1)}{4} (z_k e^{mt} - z_j e^{mt})^2 \right) dt = \\ &\frac{b(3c-1)}{4} \frac{1}{r-2m} z_i^2 + \frac{12a + b(3c-5)}{12} \frac{1}{r-m} z_i + \\ &\quad \frac{3a-b}{9} \frac{1}{r} - \frac{bc(3c-1)}{4} \frac{1}{r-2m} (z_k - z_j)^2. \end{aligned}$$

Plugging in $z_i = x_i - \frac{1}{3}$ and $m = 3k(c+1)/2$ gives

$$\begin{aligned} V_i(x) &= \int_0^\infty e^{-rt} \pi_i(\hat{y}(x(t))) dt = \\ &c_1 \left(x_i - \frac{1}{3} \right)^2 + c_2 \left(x_i - \frac{1}{3} \right) + c_3 + c_4 (x_k - x_j)^2, \quad (10) \end{aligned}$$

where

$$\begin{cases} c_1 = \frac{b(3c-1)}{4(r-3k(c+1))}, \\ c_2 = \frac{12a+b(3c-5)}{6(2r-3k(c+1))}, \\ c_3 = \frac{3a-b}{9r}, \\ c_4 = -\frac{bc(3c-1)}{4(r-3k(c+1))}. \end{cases}$$

Now consider the Hamilton-Jacobi-Bellman equations:

$$\hat{y}_i(x) \in \text{Arg} \max_{y_i \in Y_i(x)} \left(\pi_i(y_i, \hat{y}_{-i}(x)) + \sum_j \frac{\partial V_i(x)}{\partial x_j} f_j(y_i, \hat{y}_{-i}(x)) \right), \quad (11)$$

$$rV_i(x) = \pi_i(\hat{y}(x)) + \sum_j \frac{\partial V_i(x)}{\partial x_j} f_j(\hat{y}(x)). \quad (12)$$

If these equations are satisfied for all $x \in X$, then \hat{y}_i is the best response to \hat{y}_{-i} (when the strategies of player i are limited to class \mathcal{A} so that $x(t)$ never leaves X)—see Dockner et al. (2000, chapters 3 and 4).

Eq. (12) is automatically satisfied given the way V is constructed. We now check Eq. (11). Let

$$g(y_i, x) = \pi_i(y_i, \hat{y}_{-i}(x)) + \sum_j \frac{\partial V_i(x)}{\partial x_j} f_j(y_i, \hat{y}_{-i}(x)).$$

Using (7), (10) and the definitions for π_i , f_i to expand $g(y_i, x)$ and maximising the result w.r.t. $y_{ij} + y_{ik} = x_i$ gives us

$$\tilde{y}_{ij}(x) = \frac{x_i + d(x_k - x_j)}{2}, \quad (13)$$

$$d = \frac{1-c}{4} - \frac{ck(3c-1)}{2(r-3k(c+1))}. \quad (14)$$

In principle, it is possible to obtain a corner solution when maximising $g(y_i, x)$; however, it is never the case for $x \in X$.

Strategy \hat{y}_i is the best response strategy if (7) coincides with (13), i.e., if $c = d$. Using (14) to expand the equation $c = d$ and simplifying it yields

$$9c^2 + \left(14 - 5\frac{r}{k}\right)c + \left(\frac{r}{k} - 3\right) = 0. \quad (15)$$

Eq. (15) has real roots only when $\frac{r}{k} \in (-\infty, 76/25] \cup [4, \infty)$. It is straightforward to show the following. If $\frac{r}{k} \geq 4$, then both roots are strictly positive.

If $\frac{r}{k} \leq 76/25$, then one of the roots is always strictly positive, while the other root,

$$c^* = \frac{1}{18} \left(5\frac{r}{k} - 14 - \sqrt{25\left(\frac{r}{k}\right)^2 - 176\frac{r}{k} + 304} \right),$$

satisfies $c^* < -1$ if and only if $\frac{r}{k} < \frac{4}{3}$.⁸

So, if $\frac{r}{k} < \frac{4}{3}$, then there is a unique $c < -1$, defined by $c = c^*$, such that (7) constitutes an MPE. If $\frac{r}{k} \geq \frac{4}{3}$, then no such c exists. To finish the proof, it then only remains to show that (7), with $c = c^*$, constitutes an equilibrium also in class \mathcal{B} .

Fix c at c^* and consider an arbitrary strategy $\hat{y}_i(x) \in \mathcal{B}$. Under a class \mathcal{B} strategy, the game ends at some T (which is determined by $\hat{y}_i(x)$). Let

$$y_i^n(x, t) = \begin{cases} \hat{y}_i(x) & \text{if } t \leq T - \epsilon_n, \\ \hat{y}_i(x) & \text{if } t > T - \epsilon_n, \end{cases}$$

where ϵ_n is a sequence, $\epsilon_n > 0$ and $\lim_{n \rightarrow \infty} \epsilon_n = 0$. This strategy $y_i^n(x, t)$ belongs to \mathcal{A} ; therefore, it gives the same or a lower profits than the best response strategy $\hat{y}_i(x)$, i.e.

$$\begin{aligned} \int_0^\infty e^{-rt} \pi_i(\hat{y}_i(x(t))) dt &\geq \int_0^\infty e^{-rt} \pi_i(y_i^n(x(t))) dt = \\ &\int_0^{T-\epsilon_n} e^{-rt} \pi_i(\hat{y}_i(x(t))) dt + \int_{T-\epsilon_n}^\infty e^{-rt} \pi_i(\hat{y}_i(x(t))) dt. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ yields

$$\int_0^\infty e^{-rt} \pi_i(\hat{y}_i(x(t))) dt \geq \int_0^T e^{-rt} \pi_i(\hat{y}_i(x(t))) dt + V_i(x(T)).$$

On the other hand, the profits from employing strategy $\hat{y}_i(x)$ is

$$\int_0^T e^{-rt} \pi_i(\hat{y}_i(x(t))) dt + S_i(x(T)).$$

Therefore, if $S_i(x(T)) \leq V_i(x(T))$, then \hat{y}_i is the optimal strategy in class \mathcal{B} as well.

Eq. (8), as derived from from (2), (7) and $\sum_i x_i(t) = 1$, still holds even if player i has an arbitrary strategy $\hat{y}_i(x)$ (but given that other players have equilibrium strategies). Consequently, (9) holds as well.

As $x(0) \in X$, it follows from the definition of X that $x_i(0) < \frac{2}{5}$. Given (9) and using $c < -1$, we then also have that $x(T) < \frac{2}{5}$. At the same time, $x(T)$ belongs to the boundary of X . So, if it were true that $x_i(T) > x_j(T)$

⁸The fact that we get rational numbers here (76/5, 4, 4/3) is a peculiar coincidence, and it came as a surprise to the authors.

for all $j \neq i$, then it would also be true that $x_i(T) = \frac{2}{5}$. Because it is not, we have established that $x_i(T) \leq x_j(T)$ for at least some $j \neq i$. Therefore, $S_i(x(T)) = 0$. At the same time, based on $\varphi(\hat{y}_{ij}(x), \hat{y}_{ji}(x)) > 0$, we have that $V_i(x(T)) > 0$.

Thus, $S_i(x(T)) \leq V_i(x(T))$ and $\hat{y}_i(x)$, with $c = c^*$, is the best response in class \mathcal{B} as well.

Essentially, a weaker player can choose a strategy that will lead him to reach the boundary of X , but it is not optimal for him to do so. The strongest player may prefer to reach the boundary if he is still the strongest player when he does so, but he cannot achieve such results if his rivals are using equilibrium strategies. \square

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